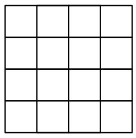
MAS114: Semester 2 Problem Booklet

1 Orbits, Functions and Symmetries

1. (Homework problem) A hotel uses electronic room keys in the form of plastic squares made by gluing together 16 small squares which may be white or black, in the pattern shown. The keys can be rotated or turned over before inserting in the reading device on the room door.



   (a) The key to an ensuite single room has 14 white squares and two adjacent black squares, forming a rectangle which may be aligned horizontally or vertically.

   What is the maximum number of ensuite single rooms so that every room has a different key (given that two keys are the same when one can be rotated or turned over to give the other)? Give the reasoning behind your answer.

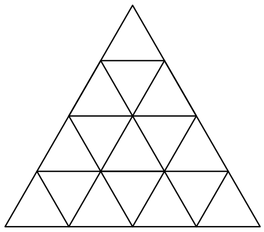
   (b) The key to an ensuite double room has 13 white squares and three black squares, forming a 3×1 or 1×3 rectangle.

   What is the maximum number of ensuite double rooms so that every room has a different key? Give the reasoning behind your answer.

   (c) The key to an ensuite family room has 12 white squares and four black squares, forming a shape as shown (or a rotation or reflection of it).

   What is the maximum number of ensuite family rooms so that every room has a different key? Give the reasoning behind your answer.

2. The diagram represents a clear glass equilateral triangular tile, patterned with a grid made up of 16 small equilateral triangles. Within the grid there are equilateral triangles made up of 1, 4, 9 or 16 small triangles. One of these is to be coloured blue.



   (a) How many equilateral triangles made up of 4 small triangles are there? How many equilateral triangles made up of 9 small triangles are there? (Do not take account of symmetry at this stage.)

   (b) Complete the following table showing the number of orbits of tiles with a blue equilateral triangle, which may be made up of $1, 4, 9$ or 16 small triangles. The layout of the table is similar to that of the tables in Problem 1.1 of the notes.

j small triangles	no. of orbits	no. of elements in each orbit	total
$j = 1$			16
$j = 4$			
$j = 9$			
$j = 16$	1	1	1
total		—	

3. For a real number a, let $f_a : \mathbb{R} \to \mathbb{R}$ and $g_a : \mathbb{R} \to \mathbb{R}$ be the functions given by $f_a(x) = a + x$ and $g_a(x) = a - x$. Simplify each of the following compositions.

   (a) $f_{50}f_{100}$; (b) $f_{50}g_{100}$; (c) $g_{50}f_{100}$; (d) $g_{50}g_{100}$.

4. Let $\alpha, \beta, \gamma \in \mathbb{R}$. Using the rot/ref formulae from the notes, express each of the following as a single rotation or reflection:

   (a) $(\text{rot}_\alpha \, \text{rot}_\beta) \, \text{rot}_\gamma$;

   (b) $(\text{ref}_\alpha \, \text{ref}_\beta) \, \text{ref}_\gamma$;

   (c) $(\text{ref}_\alpha \, \text{rot}_\beta) \, \text{ref}_\alpha$;

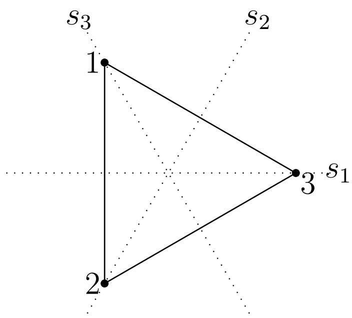
   (d) $(\text{rot}_\alpha \, \text{ref}_\beta) \, \text{rot}_\alpha$.

5. (Homework problem) Let $\alpha, \beta \in \mathbb{R}$.

   (a) Express $\mathrm{rot}_\alpha \mathrm{ref}_\alpha \mathrm{rot}_{-\alpha}$ as a single reflection.

   (b) Show that $\mathrm{rot}_\alpha \mathrm{ref}_\alpha \mathrm{rot}_{-\alpha} = \mathrm{ref}_\alpha$ if and only if $\alpha = n\pi$ for some $n \in \mathbb{Z}$. (Note: there are two direction to prove here!)

6. The equilateral triangle shown has 6 symmetries which form a group called D_3. Complete the Cayley table for this group (rotations are anticlockwise).



$e = \mathrm{rot}_0$
$r_1 = \mathrm{rot}_{\frac{2\pi}{3}}$
$r_2 = \mathrm{rot}_{\frac{4\pi}{3}}$
$s_1 = \mathrm{ref}_0$
$s_2 = \mathrm{ref}_{\frac{2\pi}{3}}$
$s_3 = \mathrm{ref}_{\frac{4\pi}{3}}$

D_3	e	r_1	r_2	s_1	s_2	s_3
e						
r_1					s_3	
r_2						
s_1						
s_2						
s_3						

(Remember that to compute fg, first apply g then f. The answer goes in the row marked f and the column marked g.)

7. The diagrams show all the eight different functions from $\{1, 2, 3\}$ to $\{1, 2\}$. For example, the first box shows the function $f : \{1, 2, 3\} \to \{1, 2\}$ such that $f(1) = f(2) = f(3) = 1$. How many of these eight functions are surjective? How many are injective?

$1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 1$	$1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$	$1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 1$	$1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 2$
$1 \mapsto 2$ $2 \mapsto 1$ $3 \mapsto 1$	$1 \mapsto 2$ $2 \mapsto 1$ $3 \mapsto 2$	$1 \mapsto 2$ $2 \mapsto 2$ $3 \mapsto 1$	$1 \mapsto 2$ $2 \mapsto 2$ $3 \mapsto 2$

8. How many different functions are there from $\{1, 2\}$ to $\{1, 2, 3\}$? How many of these are surjective? How many are injective?

9. (Homework problem)

   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $f(x, y) = (x + y, x - y)$ for all $(x, y) \in \mathbb{R}^2$. Show that f is bijective and find its inverse $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

   (b) Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $h(x, y) = (2x, xy)$ for all $(x, y) \in \mathbb{R}^2$. Show that h is neither injective nor surjective.

10. A *Gaussian integer* is a complex number of the form $a + ib$ where a and b are integers. The set of all Gaussian integers is written $\mathbb{Z}[i]$.

    (a) Draw an Argand diagram showing all Gaussian integers $a + ib$ with $-2 \leq a \leq 2$ and $-2 \leq b \leq 2$.

    (b) Show that $\mathbb{Z}[i]$ is countable. (Hint: you'll need to use a method similar to that used in the notes for $\mathbb{Q}$, indicating a "route" through $\mathbb{Z}[i]$ on your diagram and thus expressing $\mathbb{Z}[i]$ in the form $\{z_1, z_2, z_3, \ldots\}$, giving the terms as far as z_{10}.)

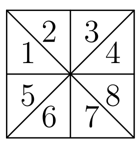
11. (Homework problem)

    (a) Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ be countable sets. Show that the union $A \cup B$ is countable.

    (b) Let I denote the set of all irrational numbers. Show that I is not countable. (Hint: $\mathbb{R} = \mathbb{Q} \cup I$. Suppose that I is countable and use (a) and what you know about $\mathbb{Q}$ and $\mathbb{R}$ to obtain a contradiction.)

2 Permutations

1. Each symmetry of the square permutes the 8 triangles shown. Write down the permutation α performed by s_1 (reflection in the x-axis) and β performed by r_1 (anticlockwise rotation through $\pi/2$).



2. Let $\alpha = (1\,2\,4)$, $\beta = (1\,3\,4)$. Write α, β and $\alpha\beta$ in two-row notation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ ? & ? & ? & ? \end{pmatrix}$.

3. For each integer k with $2 \leq k \leq 6$, write down the number of cycles of length k in S_6. Are there more permutations which are cycles in S_6 or more which are not?

4. Find the cycle decomposition of α^2 where

   (a) $\alpha = (1\ 2\ 3\ 4)$;

   (b) $\alpha = (1\ 2\ 3\ 4\ 5)$;

   (c) $\alpha = (1\ 2\ 3 \ldots 2m-1\ \ 2m)$, a cycle of even length $2m$;

   (d) $\alpha = (1\ 2\ 3 \ldots 2m\ \ 2m+1)$, a cycle of odd length $2m+1$.

5. Write the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 2 & 1 & 6 & 3 & 5 \end{pmatrix}$$

   as a product of adjacent transpositions $(a\ a+1)$ and hence decide whether α is odd or even.

   (Note: One way to do this is to first write α as a product of transpositions and then apply the formula in the notes to each transposition. However this can give very long answers so the following method is suggested.

   - STEP 1 (to get $4 \mapsto 1$): the product $p_1 = (1\ 2)(2\ 3)(3\ 4)$ sends 4 to 1.
   - STEP 2 (to get $3 \mapsto 2$): p_1 sends 3 to 4, so set $p_2 = (2\ 3)(3\ 4)p_1$. This sends 4 to 1 and 3 to 2.
   - Apply further steps to ensure $6 \mapsto 3, 1 \mapsto 4$, and so on.)

6. The formula $(b\ c) = (1\ b)(1\ c)(1\ b)$ (for b and c not equal to 1) expresses the transposition $(b\ c)$ as a product of 3 transpositions of the form $(1\ a)$ with $a \neq 1$. Using this formula, and the formula

$$(a_1\ a_2\ a_3\ \ldots\ a_k) = (a_1\ a_k)(a_1\ a_{k-1}) \ldots (a_1\ a_2),$$

   find a formula expressing a k-cycle $(a_1\ a_2 \ldots a_k)$ in which $a_i \neq 1$ for all i as a product of $k+1$ transpositions of the form $(1\ a)$ with $a \neq 1$. Express the cycle $(5\ 6\ 7)$ as such a product.

7. (a) Let $\alpha \in S_n$. Show that $\mathrm{sgn}(\alpha^2) = +1$. Deduce that there is no permutation $\alpha \in S_6$ with $\alpha^2 = (1\ 2\ 3\ 4\ 5\ 6)$.

   (b) Find $\alpha \in S_6$ such that $\alpha^2 = (1\ 2\ 3)(4\ 5\ 6)$.

   (c) Find $\alpha \in S_7$ such that $\alpha^2 = (1\ 2\ 3\ 4\ 5\ 6\ 7)$.

8. (a) Write each of the following permutations as a product of transpositions of the form $(1\ a)$ with $a \neq 1$.

      (i) The cycle $(1\ b\ c)$, where $1, b, c$ are distinct.

(ii) The permutation $\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 4 & 1 & 6 & 7 & 5 \end{pmatrix}$ (use (i) and Problem 6).

(b) Using your answers to (a), write the permutation θ as a product of cycles of the form $(1\ b\ c)$.

(c) Why can't the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 6 & 11 & 2 & 5 & 7 & 3 & 10 & 9 & 8 & 4 & 1 \end{pmatrix}$$

be written as a product of cycles of the form $(1\ b\ c)$?

9. (Homework problem) Let $\alpha, \beta \in S_{52}$ be given by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \ldots & 26 & 27 & 28 & 29 & \ldots & 52 \\ 1 & 3 & 5 & \ldots & 51 & 2 & 4 & 6 & \ldots & 52 \end{pmatrix}.$$

and

$$\beta = \begin{pmatrix} 1 & 2 & 3 & \ldots & 26 & 27 & 28 & 29 & \ldots & 52 \\ 2 & 4 & 6 & \ldots & 52 & 1 & 3 & 5 & \ldots & 51 \end{pmatrix}.$$

These arise when a pack of 52 cards is shuffled by dividing it into two packs of 26 cards which are then interlaced. (There is a choice of which pack is interlaced "above" the other.)

Which of these is a better shuffle? More precisely, if two expert shufflers repeatedly shuffle the pack, one always using α and the other always using β, for which shuffler will it take longer for the pack to revert to its original order?

10. (Homework problem) There are at least two ways of answering Question 9. One is straightforward, if a little dull. Another follows the steps below, which was the reasoning Stephen Coleman, a first-year in 2013-14, came up with to avoid having to do the dull approach. As an optional extra to the homework, can you follow the steps below and recreate Stephen's reasoning?

(a) Suppose we know that $\overline{a}^n = \overline{1}$ in $\mathbb{Z}_m$ for some $a, n, m \in \mathbb{N}$. Briefly explain why if k is the smallest positive integer for which $\overline{a}^k = \overline{1}$ in $\mathbb{Z}_m$ then $k|n$. (We will prove this formally later in the course.)

(b) What is the smallest value of n for which $\overline{2}^n = \overline{1}$ in $\mathbb{Z}_{53}$? (Hint: a little theorem from Semester 1, and part (a), should help here.)

(c) (i) By comparing $\overline{2}^n$ in $\mathbb{Z}_{53}$ with $\beta^n(1)$, find the order of β.

(ii) The shuffling represented by α is similar to the shuffling represented by β restricted to a pack of 50 cards. Explain why, and, by finding the smallest n for which $\overline{2}^n = \overline{1}$ in $\mathbb{Z}_{51}$, determine the order of α.

(iii) If two expert shufflers repeatedly shuffle the pack, one always using α and the other always using β, for which shuffler will it take longer for the pack to revert to its original order? (Hint: this problem is all about calculating orders of permutations.)

3 Groups and Subgroups

1. Complete the following table for multiplication modulo 8.

$\times$mod8	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$
$\overline{1}$				
$\overline{3}$				
$\overline{5}$				
$\overline{7}$				

Comment on how this shows that if $G = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$ then G is closed under multiplication modulo 8, that G has a neutral element and that each element of G has an inverse in G. Is this enough to mean that $(G, \times\text{mod } 8)$ is a group?

2. In each of the following you are given two elements a, b of a group G. In each case, find x and y in G such that $ax = b$ and $b = ya$.

   (a) G is the orthogonal group O_2; $a = \text{rot}_{\frac{\pi}{6}}, b = \text{ref}_{\frac{\pi}{2}}$.

   (b) G is the symmetric group S_4; $a = (1\ 2\ 4), b = (1\ 3)$.

   (c) G is $\mathbb{Z}_7 \backslash \{\overline{0}\}$ under multiplication modulo 7, $a = \overline{3}$ and $b = \overline{4}$.

3. Let G be a group and let $g \in G$. Show that if $g^2 = g$ then $g = e$.

4. (Homework problem) Let G be a group and let $g, h, k \in G$.

   (a) Show that if $ghg = gkg$ then $h = k$.

   (b) Show that if G is abelian and $hgh = kgk$ then $h^2 = k^2$.

   (c) Using the Cayley table for D_3 (see Problem 6 from Chapter 1), find three elements $g, h, k \in D_3$ such that $hgh = kgk$ but $h^2 \neq k^2$.

5. (Homework problem) Let G be a group in which every element is its own inverse, that is $a^{-1} = a$ for all $a \in G$. By considering the inverse of a product of two elements of G, show that G is abelian.

6. The six elements of S_3 are id, $\rho_1 = (1\ 2\ 3)$, $\rho_2 = (1\ 3\ 2)$, $\sigma_1 = (1\ 2)$, $\sigma_2 = (1\ 3)$ and $\sigma_3 = (2\ 3)$. Below is a partially completed Cayley table for S_3.

(a) Enter the products $\rho_1\sigma_1$, $\sigma_1\rho_1$ and $\sigma_1\sigma_2$.

(b) Use the Latin square property and the rules for combining odd and even permutations to complete the table without further calculation of products. (Note: $\mathrm{id}, \rho_1, \rho_2$ are even, each σ_i is odd).

S_3	id	ρ_1	ρ_2	σ_1	σ_2	σ_3
id	id	ρ_1	ρ_2	σ_1	σ_2	σ_3
ρ_1	ρ_1		id			
ρ_2	ρ_2	id				
σ_1	σ_1			id		
σ_2	σ_2				id	
σ_3	σ_3					id

7. (Homework problem) Let H be the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}, \qquad a, b \in \mathbb{R},\ b \neq 0.$$

(a) Using determinants, show that every element of H is invertible and hence that $H \subseteq GL_2(\mathbb{R})$.

(b) Show that H is a subgroup of $GL_2(\mathbb{R})$. Is H abelian?

8. Consider the 26 capital letters of the alphabet, each written as symmetrically as possible. For example

HIL NOQX Y

State the letters with

(a) group of symmetries of order 1 (the only symmetry is the identity);

(b) group of symmetries of order 2 consisting of the identity and a reflection;

(c) group of symmetries of order 2 consisting of the identity and a rotation;

(d) group of symmetries of order 4;

(e) group of symmetries of order 6;

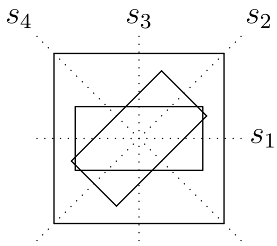
(f) group of symmetries of order 8;

(g) infinite group of symmetries.

9. (a) Let K be Klein's 4-group, the group of symmetries of a non-square rectangle. Each element of K performs a permutation of the vertices of the rectangle, numbered as shown below. The four permutations which arise in this way form a subgroup of S_4 isomorphic to K. Write down the four elements of this subgroup.

(b) For each of the two non-square rectangles in the diagram below, write down the elements of D_4 which are symmetries of the rectangle.

(This gives two subgroups of D_4 which are isomorphic to Klein's 4-group.)



10. Let G be the group $U_2 = \{1, -1\}$ under multiplication. Complete the given Cayley table for the direct product $G \times G$.

$G \times G$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,1)$				
$(1,-1)$				
$(-1,1)$				
$(-1,-1)$				

11. Show that the following direct products all have the same order:

$$D_4 \times D_6, \quad S_4 \times K, \quad U_3 \times U_{32}.$$

(D_n is the group of symmetries of a regular n-gon, S_n is the group of permutations of $\{1, 2, \ldots, n\}$, K is Klein's 4-group and U_n is the group of nth roots of unity.)

12. Two of the groups with tables appearing in the earlier problems are isomorphic to Klein's 4-group. Which are they?

4 Cyclic Groups

1. (a) Let p be a prime number and let $0 < a < p$ be a positive integer. Use Fermat's Little Theorem, along with a result from the notes, to deduce that the order of $\overline{a}$ in $\mathbb{Z}_p \backslash \{\overline{0}\}$ is a factor of $p - 1$.

   (b) For which of the following primes p does $\overline{2}$ have order $p - 1$ in $\mathbb{Z}_p \backslash \{\overline{0}\}$ (and hence is a generator for the group)?

$$11, 13, 17, 31, 41.$$

If $\overline{2}$ does not generate the group, find an element $\overline{a}$ of $\mathbb{Z}_p \backslash \{\overline{0}\}$ which does. (For the larger primes, this may be slow without a computer. Those who know Python may manage to write a script to investigate this. Those who don't might manage to get somewhere with a spreadsheet.)

2. (Homework problem) Let $G = \langle g \rangle$ be a cyclic group of order $n > 1$ generated by g.

   (a) What is the order of $G \times G$?

   (b) Let $(a, b) \in G \times G$. Show that $(a, b)^n = (e, e)$.

   (c) Does $G \times G$ contain any elements of order n^2? Is $G \times G$ cyclic?

3. In the orthogonal group O_2, let $g = \mathrm{rot}_{\frac{2\pi}{27}}$.

   (a) Write down the orders of g, g^9 and g^{12}.

   (b) List all the elements of the cyclic subgroups $\langle g^9 \rangle$ and $\langle g^{12} \rangle$ (either in the form g^d, $0 \leq d \leq 26$, or rot_θ).

   (c) Write down integers d and e such that $0 \leq d, e \leq 26$, $d \neq 9$, $e \neq 12$, g^d has the same order as g^9 and g^e has the same order as g^{12}.

4. (a) The rotation group of the circle, SO_2, has exactly one element of order 2. What is it?

   (b) In each of the following groups, how many elements of order 2 are there?

      (i) O_2 (the group of symmetries of the circle);

      (ii) D_5 (the group of symmetries of the regular pentagon);

      (iii) D_6 (the group of symmetries of the regular hexagon).

5. How many different *cyclic* subgroups are there in D_5 (the group of symmetries of the regular pentagon)?

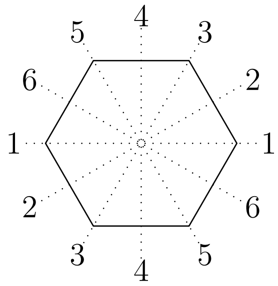
6. Let G be the rotation group of a regular octagon. This is a cyclic group of order 8 generated by $g = \mathrm{rot}_{\frac{\pi}{4}}$, so its subgroups are all cyclic. How many distinct subgroups does it have? List these subgroups and, for each of them, list all its elements in the form g^j.

7. How many different subgroups are there in U_{24}, the group of 24th roots of unity?

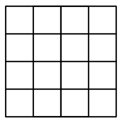
5 Group Actions

1. The group D_6 of symmetries of the regular hexagon acts on the six numbered axes of symmetry below; for example $r_1 * 1 = 3$. List all the elements of the orbit $\mathrm{orb}(1)$ and all the elements of the stabilizer $\mathrm{stab}(1)$.

(D_6 consists of 6 rotations $e = r_0, \ldots, r_5$ and 6 reflections s_j (for $j = 1, \ldots, 6$) in the axis numbered j.)



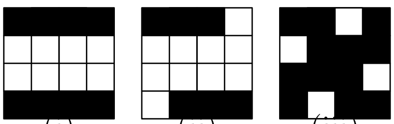
2. The group D_4 of symmetries of the square acts on the set X of colourings of the pattern



with 2 colours, black and white. For each of the shown colourings x, write down

(a) the number of elements, $|\operatorname{orb}(x)|$, in the orbit of x;

(b) all elements of the stabilizer $\operatorname{stab}(x)$.



(i) (ii) (iii)

3. (Homework problem) Let S_n act on the collection of polynomials in $x_1, x_2, \ldots, x_n$. Such a polynomial p is called *symmetric* if $\operatorname{stab}(p) = S_n$ (that is, if every permutation of its variables leaves it unchanged).

(a) If p is a symmetric polynomial, what is $\operatorname{orb}(p)$?

(b) It can be shown that every symmetric polynomial in $x_1, \ldots, x_n$ can be be written as a sum of products of *elementary symmetric polynomials*. When $n = 3$, the elementary symmetric polynomials are

$$\begin{aligned} e_0 &= 1 \\ e_1 &= x_1 + x_2 + x_3 \\ e_2 &= x_1x_2 + x_2x_3 + x_3x_1 \\ e_3 &= x_1x_2x_3. \end{aligned}$$

For example,

$$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 = x_1x_2x_3(x_1 + x_2 + x_3) = e_3e_1.$$

Write each of the following symmetric polynomials in terms of the elementary symmetric polynomials $e_0, \ldots, e_3$ above.

(i) $x_1^2 + x_2^2 + x_3^2$.

(ii) $x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$.

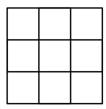
(iii) $x_1^3 + x_2^3 + x_3^3$.

4. In how many essentially different ways can the square tile shown below be coloured

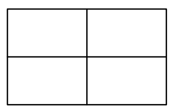
   (a) using n colours when the tile is a ceramic floor tile and cannot be turned over;

   (b) using n colours when the tile is made of glass and can be turned over;

   (c) with 6 orange regions and 3 yellow regions when the tile is made of glass and can be turned over?



5. In how many essentially different ways can the shown rectangular glass tile, which is not square and can be turned over, be coloured using n colours?



6. For the action of S_3 on polynomials in x_1, x_2, x_3, write down all the permutations in $\mathrm{stab}(p)$ and all the polynomials in $\mathrm{orb}(p)$ for each of the following polynomials p:

$$\text{(i)} \quad p = x_1x_2 + x_3^2, \quad \text{(ii)} \quad p = x_1^2x_2 + x_2^2x_3 + x_3^2x_1.$$

6 Equivalence Relations

1. Let $\sim$ be the relation on $\mathbb{R}$ given by

$$a \sim b \iff a - 2b \in \mathbb{Z}.$$

Show that $\sim$ is not reflexive, not symmetric and not transitive.

2. Let $\sim$ be the relation on $\mathbb{C}$ given by

$$a \sim b \iff a - b \in \mathbb{R}.$$

Show that $\sim$ is an equivalence relation, and list three different elements of the equivalence class of $11 + 4i$.

7 Cosets and Lagrange's Theorem

1. In each of the following, you are given a finite group G and a subgroup H of G. In each case, find all the distinct left cosets of H in G.

   (a) $G = D_3$ and $H = \{e, s_2\}$, the cyclic subgroup generated by the reflection s_2. (The Cayley table for D_3 is given below.)

   (b) $G = D_3$ and $H = \{e, r_1, r_2\}$, the cyclic subgroup generated by r_1.

   (c) $G = \langle g \rangle$ is a cyclic group of order 12 and $H = \{e, g^3, g^6, g^9\}$, the cyclic subgroup generated by g^3.

D_3	e	r_1	r_2	s_1	s_2	s_3
e	e	r_1	r_2	s_1	s_2	s_3
r_1	r_1	r_2	e	s_2	s_3	s_1
r_2	r_2	e	r_1	s_3	s_1	s_2
s_1	s_1	s_3	s_2	e	r_2	r_1
s_2	s_2	s_1	s_3	r_1	e	r_2
s_3	s_3	s_2	s_1	r_2	r_1	e

2. Give a reason why the symmetric group S_4 cannot have a subgroup of order 5. Give an example of a subgroup of S_5 of order 6.

3. Let $G = \langle g \rangle$ be a cyclic group of order 14. For each of the positive integers d listed below, either write down a subgroup H of G of order d, or a subgroup of the symmetric group S_4 of order d, or explain why no such subgroup exists.

$$2, 5, 7, 11.$$

4. Let G be a group of order 77 and let H, K be nontrivial proper subgroups of G with $|H| > |K|$.

   (a) Write down $|H|$ and $|K|$.

   (b) Show that the only element of G in both H and K is e.

5. Use Fermat's Little Theorem to find the remainders when 43^{462} is divided by 19 and by 47.

6. Using Lagrange's Theorem, show that every proper subgroup of a group of order 6 is cyclic. Hence find all the different subgroups of the symmetric group S_3.

8 Orbits and Stabilizers

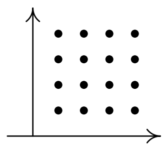
1. For the action of S_4 on polynomials in x_1, x_2, x_3, x_4, find $|\operatorname{stab}(p)|$ and $|\operatorname{orb}(p)|$ for each of the following polynomials p. (Remember that $|\operatorname{orb}(p)||\operatorname{stab}(p)|$ must be $|S_4| = 24$.)

$$\text{(i)} \quad p = x_1x_2x_3 + x_4, \qquad \text{(ii)} \quad p = x_1x_2 + x_3 + x_4.$$

2. The group S_4 acts on the set $X = \{1, 2, 3, 4\}$ by the rule $\alpha * x = \alpha(x)$ (for all $\alpha \in S_4$ and $x \in X$). Write down $|\operatorname{orb}(1)|$ and $|\operatorname{stab}(1)|$.

   List all the elements of the set $\operatorname{send}_1(2)$ and hence, or otherwise, find all elements of the left coset αH where $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $H = \operatorname{stab}(1)$.

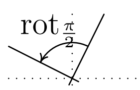
3. Let X be the set of 16 points $(x, y) \in \mathbb{R}^2$ with $x, y \in \mathbb{Z}$ and $1 \le x, y \le 4$.



   The group S_4 acts on this set by the rule $\alpha * (x, y) = (\alpha(x), \alpha(y))$ (for all $\alpha \in S_4$ and $(x, y) \in X$). For example $(1\ 2\ 3) * (2, 1) = (3, 2)$. For each of the following points P, find $|\operatorname{stab}(P)|$ and $|\operatorname{orb}(P)|$. (Remember that $|\operatorname{orb}(P)||\operatorname{stab}(P)|$ must be $|S_4| = 24$.)

$$\text{(i)} \quad P = (1, 2), \qquad \text{(ii)} \quad P = (1, 1).$$

4. The orthogonal group O_2 acts on the set of all straight lines through the origin. For example, $\operatorname{rot}_{\frac{\pi}{2}}$ sends the line $y = 2x$ to the line $y = \frac{-1}{2}x$.



   (a) The stabilizer of the x-axis for this action has 4 elements. What are they?

   (b) How many elements of O_2 send the x-axis to the y-axis? List them.

5. Let $G = GL_2(\mathbb{R})$, the group of all invertible 2×2 real matrices. Then G acts on $\mathbb{R}^2$ by the rule $A * v = Av$ for all $A \in G$ and $v \in \mathbb{R}^2$, where elements of $\mathbb{R}^2$ are written as 2×1 matrices.

   (a) Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and let H be the set of 2×2 real matrices of the form

   $$\begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}, \qquad a, b \in \mathbb{R}, a - b \neq 0.$$

   Show that $H = \text{stab}(v)$ and hence that H is a subgroup of G.

   (b) Find the stabilizer of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.