The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Numbers and Groups **2 hours**

Answer ***all questions****.*
You should justify your answers carefully unless the question states otherwise.

1 (i) Showing all your working, find any integer satisfying the simultaneous linear congruences

$$x \equiv 57 \pmod{103},$$
$$x \equiv 24 \pmod{140}.$$

(4 marks)

(ii) Why does each of the following equations have no solutions in integers x and y?

(a) $6x + 9y = 100$;

(b) $2x^2 + 1 = 6y^2$;

(c) $7x^2 + y^2 = 70003$. ***(4 marks)***

(iii) By raising both sides to the fifth power, or otherwise, find all solutions to

$$x^5 \equiv 11 \pmod{13}.$$

Explain all your working. ***(2 marks)***

2 (i) (a) What does it mean for a sequence $a_1, a_2, \ldots$ to *converge* to a limit ℓ? ***(2 marks)***

(b) Explain briefly why the sequence $r_n = \dfrac{4^n}{2^n + 1}$ does not converge. ***(2 marks)***

(c) Show, however, that the sequence $s_n = \dfrac{2^n}{2^n + 1}$ does converge, stating its limit clearly. ***(4 marks)***

(ii) What does it mean for a sequence $a_1, a_2, \ldots$ to be a Cauchy sequence? Is there a sequence which is convergent but not Cauchy? ***(2 marks)***

3 In this question, define the function L from the positive integers to the positive integers by taking

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 3.$$

(i) Is L injective? Is L surjective? Is L bijective? Give brief reasons why. ***(4 marks)***

(ii) Show by strong induction on n that $L_n = F_{n-1} + F_{n+1}$ for all integers $n \geq 1$, where F_n is the Fibonacci sequence defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

(3 marks)

(iii) Show that there is no n for which $5 \mid L_n$. ***(3 marks)***

4 (i) Let $n \geq 2$ be a positive integer. Define what is meant by a permutation on the set $\{1, \ldots, n\}$ and what is meant by the order of such a permutation. ***(2 marks)***

(ii) In S_9, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 5 & 3 & 4 & 1 & 2 & 7 & 8 & 9 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 3 & 9 & 4 & 6 & 2 & 8 & 7 \end{pmatrix}.$$

Find the cycle decompositions of α^2 and β^2. By writing them as a product of transpositions, or otherwise, determine their parity (odd or even). ***(4 marks)***

(iii) A permutation is applied twice to the string SUPERBHAT. The second output is THRBAEPUS. There are two possibilities for the first output. What are they?

(Note: it is sufficient to find the two possible first outputs. You do not need to show that they are the only possibilities.) ***(3 marks)***

(iv) How many permutations $\alpha \in S_9$ are there with $\alpha^2 = (1\ 2\ 3\ 4)$? ***(1 mark)***

5 (i) As usual, let $GL_2(\mathbb{R})$ be the group of 2×2 invertible matrices with real entries under matrix multiplication.

(a) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. Show that $A = A^{-1}$ and $B = B^{-1}$. ***(2 marks)***

(b) For each set below, determine whether or not it is a subgroup of $GL_2(\mathbb{R})$, justifying your answers.

You may use, without proof, the facts that $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.

$$\begin{aligned} H_1 &= \{A \in GL_2(\mathbb{R}) : \det(A) = 1\}; \\ H_2 &= \{A \in GL_2(\mathbb{R}) : \det(A) = 0\}; \\ H_3 &= \{A \in GL_2(\mathbb{R}) : A = A^{-1}\}. \end{aligned}$$

(6 marks)

(ii) Let G be any group of order 17.

(a) Show that if $g \in G$ is not the identity element, then g has order 17.

(b) Show that G has precisely 2 subgroups.

(2 marks)

6 (i) Let G be a group acting on a non-empty set X. Let $x, y \in X$ and $g \in G$.

For each statement below, determine whether it is true or false. (Note: you must justify your answers to get the marks.)

(a) $\text{fix}(g) = \{x \in X : g * x = x\}$.

(b) $\text{stab}(x)$ contains the identity element from G.

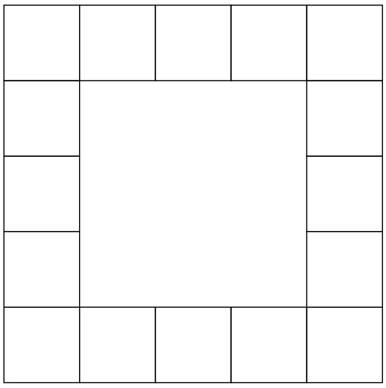
(c) $\text{stab}(x) = \{y \in X : y = g * x \text{ for some } g \in G\}$.

(d) If $y \in \text{orb}(x)$ then $x \in \text{orb}(y)$.

(e) $\sum_{g \in G} |\,\text{fix}(g)| = \sum_{x \in X} |\,\text{stab}(x)|$.

(5 marks)

(ii) A stained-glass frame, which can be turned over, is to be formed from 16 small coloured glass squares stuck together in the shape below.



Find the number of essentially different frames that can be made using 8 green and 8 blue squares, making your method clear. ***(5 marks)***

End of Question Paper