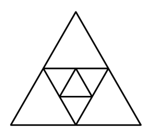
ORBIT COUNTING THEOREM

5 minute review. Briefly recap the definitions of orb(x), stab(x), send$_x(y)$ and fix(g) for a group G acting on a non-empty set X, then state the Orbit Counting Theorem (see attached sheet).

Class warm-up. How many different finished noughts and crosses grids are there? Ask for suggestions on how to approach this problem, then let people work on it (Q1 below).

Problems. Choose from the below.

1. How many essentially different ways can a noughts and crosses grid be filled in with 5 crosses and 4 noughts? Two grids should be considered the same if one can be rotated or reflected to give the other.
2. In how many essentially different ways can the equilateral triangular tile below, which can be turned over, be coloured
   (a) using n colours;
   (b) so that there are 4 blue regions and 3 green regions?



3. The group D_4 has ten subgroups: one is D_4, seven of them are cyclic and two are isomorphic to Klein's 4-group.
   (a) Identify these subgroups.
   (b) D_4 acts on the set of colourings of a 5×5 grid in two colours. For each of the subgroups in part (a), identify a colouring for which that subgroup is the stabilizer.
4. Let S_n act on the set of polynomials in n variables in the usual way, where $n \geq 3$. For each subset of S_n below, write down a polynomial with that subgroup as a stabilizer or explain why no such polynomial is possible.
   (a) S_n;
   (b) $\{\text{id}\}$;
   (c) the set of all even permutations;
   (d) the set of all odd permutations;
   (e) $\{\alpha \in S_n : \alpha(n) = n\}$.

Homework. None

Selected answers and hints.

1. Let D_4 act on the set of 3×3 grids filled with five $\times$s and four 0s. From the diagrams below, we find that $|\operatorname{fix}(e)| = \binom{9}{4} = 126$, $|\operatorname{fix}(r_1)| = |\operatorname{fix}(r_3)| = 2$ (one of A or B is 0s, the rest $\times$s), $|\operatorname{fix}(r_2)| = \binom{4}{2} = 6$ (two of A–D are 0s, the rest $\times$s), $|\operatorname{fix}(s_1)| = |\operatorname{fix}(s_3)| = 3 + 3 \times 3 = 12$ (two of A–C or one of A–C and two of D–F are 0s), $|\operatorname{fix}(s_2)| = |\operatorname{fix}(s_4)| = 12$ (same reasoning as for s_1). This, by the Orbit Counting Theorem, the number of essentially different grids is $\frac{1}{8}(126 + 2 \times 2 + 6 + 4 \times 12) = 23$.

   Of course, the number calculated above is not the same as the number of essentially different *games* of noughts and crosses. Is that higher or lower?

   r_1, r_3

A	B	A
B	C	B
A	B	A

   r_2

C	B	A
D	E	D
A	B	C

   s_1

C	B	A
D	E	F
C	B	A

   s_2

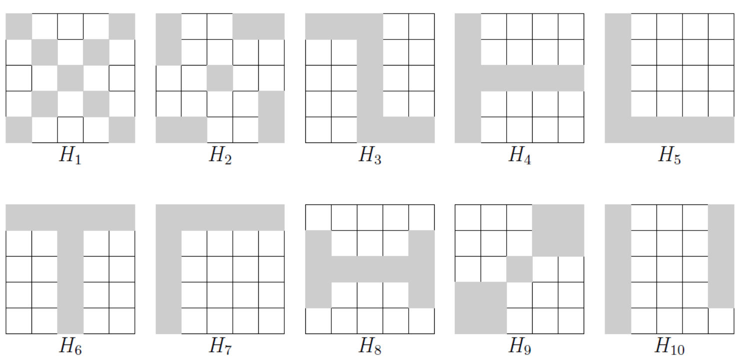
A	B	D
C	E	B
F	C	A

2. We count the number of orbits for the action of D_3 on the set of colourings.

   (a) Here, $|\operatorname{fix}(e)| = n^7$. By drawing diagrams, $|\operatorname{fix}(r_1)| = |\operatorname{fix}(r_2)| = n^3$ and $|\operatorname{fix} s_2| = |\operatorname{fix}(s_1)| = |\operatorname{fix} s_3| = n^5$. By the Orbit Counting Theorem, the number of essentially different colourings is $\frac{1}{6}(n^7 + 3n^5 + 2n^3)$.

   (b) This time, $|\operatorname{fix}(e)| = \binom{7}{3} = 35$ (choose 3 of the 7 regions to be green). We find $|\operatorname{fix}(r_1)| = |\operatorname{fix}(r_2)| = 2$ (the centre must be blue, along with one of the inner or outer ring of triangles). Also, $|\operatorname{fix}(s_1)| = |\operatorname{fix} s_2| = |\operatorname{fix} s_3| = 2 \times 3 + 1 = 7$ (the green ones being one pair of triangles off the line of symmetry and a single triangle on, or all three triangles on the line of symmetry). By the Orbit Counting Theorem, the number of essentially different colourings is $\frac{1}{6}(35 + 4 + 21) = 10$.

3. The subgroups are $H_1 = D_4$, $H_2 = \langle r_1 \rangle = \{e, r_1, r_2, r_3\}$, $H_3 = \langle r_2 \rangle = \{e, r_2\}$, $H_4 = \langle s_1 \rangle = \{e, s_1\}$, $H_5 = \langle s_2 \rangle = \{e, s_2\}$, $H_6 = \langle s_3 \rangle = \{e, s_3\}$, $H_7 = \langle s_4 \rangle = \{e, s_4\}$, $H_8 = \{e, r_2, s_1, s_3\}$, $H_9 = \{e, r_2, s_2, s_4\}$ and $H_{10} = \{e\}$. There are many possibilities for pictures; some correct options are below.



4. (a) Any symmetric polynomial, e.g. $x_1 \cdots x_n$ works here.

   (b) The polynomial $x_1 + x_1 x_2 + \ldots + x_1 \cdots x_n$ works here.

   (c) The alternating polynomial, a_n, works. Are there any others?

   (d) This is not a subgroup of S_n so can't be a stabilizer.

   (e) The polynomial $x_n(x_1 + \ldots + x_{n-1})$ works here.

For more details, start a thread on the discussion board.

Definitions. Let G be a group acting on a non-empty set X.

- For any $x \in X$, the *orbit* of x is the set
$$\operatorname{orb}(x) = \{y \in X : y = g * x \,\text{for some}\, g \in G\}.$$
- The *stabilizer* of x is the set
$$\operatorname{stab}(x) = \{g \in G : g * x = x\}.$$
- For each $y \in \operatorname{orb}(x)$, the *sending set* $\operatorname{send}_x(y)$ is given by
$$\operatorname{send}_x(y) = \{g \in G : g * x = y\}.$$
- For each $g \in G$, the *fixed set* of g is the subset
$$\operatorname{fix}(g) = \{x \in X : g * x = x\}.$$

Theorem (The orbit-counting theorem)**.** *Let G be a finite group acting on a non-empty finite set X and let n be the number of orbits. Then*
$$n = \frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|.$$