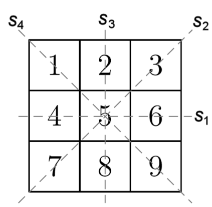
CYCLIC GROUPS AND GROUP ACTIONS

5 minute review. Briefly recap what a cyclic group is, the fact that all subgroups of cyclic groups are cyclic, and that for a finite cyclic group of order n there is one distinct subgroup for each positive divisor of n. Also cover what is meant by a group G acting on a non-empty set X (which could be illustrated by D_4 acting on the numbered corners of a square, for example), and recap the definitions of $\text{orb}(x) = \{y \in X : y = g * x \text{ for some } x \in X\}$ and $\text{stab}(x) = \{g \in G : g * x = x\}$.

Class warm-up. Let G act on a non-empty set X. Suppose G has order n. What is the maximum number of elements in $\text{orb}(x)$? What is the minimum? What about $\text{stab}(x)$? If $\text{stab}(x)$ is 'big', what is the implication for $\text{orb}(x)$? If $|\text{stab}(x)| = n$, what is $|\text{orb}(x)|$? What about if $|\text{stab}(x)| = 1$?

Problems. Choose from the below.

1. For each element of $g \in D_4$, state its order and write down the elements of $\langle g \rangle$. How many different *cyclic* subgroups are there in D_4?
2. The group D_4 acts on the numbered regions 1 to 9 below in the usual way.



   (a) What is $s_2 * 7$? What is $r_3 * 1$?

   (b) For which $g \in D_4$ is $g * 3 = 9$? For which $h \in D_4$ is $h * 6 = 5$?

   (c) How many elements $g \in D_4$ fix 1, in the sense that $g * 1 = 1$?

   (d) How many different $1 \leq y \leq 9$ are there with $y = g * 2$ for some $g \in D_4$?

   (e) What is $\text{orb}(4)$? What is $\text{stab}(5)$?

3. How many distinct subgroups does the rotation group of a regular 2^k-gon have, where $k > 1$ is a positive integer? What are these subgroups?
4. Let G act on a non-empty set X, and let $g_1, g_2 \in G$ and $x \in X$.

   (a) If $y = g_1 * x$, can you find $h \in G$ with $x = h * y$? If $z = g_2 * x$, can you find $k \in G$ with $z = k * y$?

   (b) What can you say about $\text{orb}(x)$ compared with $\text{orb}(y)$, where $y = g_1 * x$?

5. Let G be the group $(\mathbb{R}, +)$ and H be $(\mathbb{R}_{>0}, \times)$. Is the function $f : G \to H$ given by $f(x) = e^x$ an isomorphism? Can you write down an isomorphism from H to G?
6. How many 'different' groups are there are of order 4? (Here, different means non-isomorphic.)

Homework. Chapter 5, Q3

For the warm-up, $\text{orb}(x)$ can have at most n things in (if every element of G sends x to a unique new element of X) and at least 1 (as x itself will always be in $\text{orb}(x)$). Similarly, $\text{stab}(x)$ has at most n things in (when everything sends x to itself) and always has at least the identity element in (as $e * x = x$).

If $\text{stab}(x)$ has lots of things in, we'd expect $\text{orb}(x)$ to have not very many: for example, if $|\,\text{stab}(x)| = n$ then everything sends x to itself, so $|\,\text{orb}(x)| = 1$. If $|\,\text{stab}(x)| = 1$ then we'd expect $|\,\text{orb}(x)|$ to be big; later in the course we'll prove that it must have size n.

Selected answers and hints.

1. D_4 has 7 different cyclic subgroups, namely $\langle e\rangle = \{e\}$, $\langle r_1\rangle = \{e, r_1, r_2, r_3\}$, $\langle r_2\rangle = \{e, r_2\}$, $\langle s_1\rangle = \{e, s_1\}$, $\langle s_2\rangle = \{e, s_2\}$, $\langle s_3\rangle = \{e, s_3\}$ and $\langle s_4\rangle = \{s_4\}$. (Two of the 8 elements, r_1 and r_3, generate the same subgroup.)
2. (a) $s_2 * 7 = 7$ and $r_3 * 1 = 3$.
   (b) $g * 3 = 9$ when g is either of s_1 and r_3. There is no $h \in D_4$ with $h * 6 = 5$.
   (c) There are two such elements, namely e and s_4.
   (d) There are four such numbers: 2, 4, 6 and 8.
   (e) $\text{orb}(4) = \{2, 4, 6, 8\}$ and $\text{stab}(5) = D_4$.
3. As the rotation group of the regular 2^k-gon is cyclic, all its subgroups must be cyclic and there will be one for each factor of 2^k. Thus there are $k + 1$ of these. The subgroups are all of the form $\langle r_{2^j}\rangle$ for $1 \leq j \leq k$.
4. (a) If $y = g_1 * x$ then $x = g_1^{-1} * y$. If $z = g_2 * x$ then $z = (g_2 g_1^{-1}) * y$.
   (b) If $y = g_1 * x$, then $\text{orb}(x) = \text{orb}(y)$, as any element which can be reached from x can also be reached from y, and vice versa. We'll prove this formally in the lectures.
5. It is, although to see this one has to be careful with the group operations. Notice that the operation in $\mathbb{R}$ is addition whereas in $\mathbb{R}_{>0}$ it is multiplication, so to be an isomorphism we require $f(x + y) = f(x)f(y)$. This is true as $f(x + y) = e^{x+y} = e^x e^y = f(x)f(y)$. An isomorphism from H to G is given by $g(x) = \ln x$ (which works, as $g(xy) = \ln(xy) = \ln x + \ln y = g(x) + g(y)$).
6. There are only two. Try filling in a Cayley table for $\{e, a, b, c\}$. The first row and column are filled in without thought. Bearing in mind the Latin Square property, there are only two distinctly different options for a^2. From there, the rest follows.

For more details, start a thread on the discussion board.