SUBGROUPS

5 minute review. Recap that a subgroup of a group G is a subset H which is itself a group using the same binary operation. Cover the Subgroup Criterion (see final page), and give an example of how to use it, e.g. H as the positive real numbers as a subgroup of $\mathbb{R} \setminus \{0\}$ under multiplication. Remind students what is meant by (a) the order of a group, and (b) the order of an element in a group.

Class warm-up. By applying at the Subgroup Criterion, which of the sets below are subgroups of of $\mathbb{Q} \setminus \{0\}$ under multiplication?

$$H_1 = \left\{ \frac{a}{b} \in \mathbb{Q} : a \text{ and } b \text{ are odd} \right\},$$
$$H_2 = \left\{ \frac{a}{b} \in \mathbb{Q} : b \text{ is odd and } a \text{ is even and non-zero} \right\}.$$

Problems. Choose from the below.

1. Which of the sets below are subgroups of $\mathbb{R}$ under addition?
$$H_1 = \{a \in \mathbb{R} : a \geq 0\},$$
$$H_2 = \{\theta \in \mathbb{R} : \sin\theta = 0\}.$$
2. Consider addition on $\mathbb{R}^2$ given by $(a, b) + (c, d) = (a + c, b + d)$. Show that this binary operation makes $\mathbb{R}^2$ into a group. Which of the following are subgroups of $\mathbb{R}^2$ with the given binary operation?
$$H_1 = \{(x, y) \in \mathbb{R}^2 : y = 0\};$$
$$H_2 = \{(x, y) \in \mathbb{R}^2 : 3y + x + 1 = 0\};$$
$$H_3 = \{(x, y) \in \mathbb{R}^2 : 3y + x = 0\};$$
$$H_4 = \{(x, y) \in \mathbb{R}^2 : y = x \text{ or } y = -x\}.$$
(I'd recommend drawing the sets H_1–H_4 to get a feel for what's going on.)
3. Sketch a subset of $\mathbb{R}^2$ which has
   (a) a group of symmetries of order 8;
   (b) a rotation group of order 6;
   (c) a rotation group of order 3;
   (d) a group of symmetries of order 1;
   (e) a group of symmetries of order 3.
4. Find the orders of the following elements.
$$\overline{2} \text{ in } \mathbb{Z}_3 \setminus \{\overline{0}\}; \quad \overline{5} \text{ in } \mathbb{Z}_{11} \setminus \{\overline{0}\}; \quad \overline{p-1} \text{ in } \mathbb{Z}_p \setminus \{\overline{0}\}.$$
5. (a) What day of the week will it be 5^{3602} days from today?
   (b) What will the time be $13^{123456789}$ hours from now?

Homework. Chapter 3, Q7

For the warm-up, H_1 is a subgroup of $\mathbb{Q} \setminus \{0\}$ as it satisfies SG1, SG2 and SG3: $1 = \frac{1}{1} \in H_1$ so $H_1 \neq \emptyset$; if $\frac{a}{b}, \frac{c}{d} \in H_1$ with a, b, c, d all odd, then $\frac{ac}{bd} \in H_1$ as ac and bd are both odd; if $\frac{a}{b} \in H_1$ with a, b both odd, then $\frac{b}{a} \in H_1$ as b, a are both odd. On the other hand, H_2 is not a subgroup as, for example, SG3 fails: $2 = \frac{2}{1} \in H_2$, but $2^{-1} = \frac{1}{2} \notin H_2$ (can you prove this?).

Selected answers and hints.

1. Since $1 \in H_1$ but $-1 \notin H_1$, SG3 fails and H_1 is not a subgroup of $\mathbb{R}$ under addition.

   For H_2, we need to check that SG1,2,3 hold: SG1 holds because $\sin 0 = 0$ so $0 \in H_2$ and therefore $H_2 \neq \emptyset$. For SG2, let $\theta, \phi \in H_2$. Then $\sin\theta = \sin\phi = 0$. Hence $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = 0$ so $\theta + \phi \in H_2$. For SG3, if $\theta \in H_2$ then $\sin\theta = 0$, so $\sin(-\theta) = -\sin\theta = 0$, so $-\theta \in H_2$. By the additive version of the subgroup criterion, H_2 is a subgroup of $\mathbb{R}$. (Alternatively, H_2 consists of all integer multiples of π, and one can proceed from there.)

2. To show that $\mathbb{R}^2$ is a group under addition requires checking the group axioms, G1-4. Briefly, $\mathbb{R}^2$ is clearly closed under addition, associativity follows from the associativity of addition of real numbers, the neutral element is $(0, 0)$ and the inverse of (a, b) is $(-a, -b)$.

   H_1 and H_3 are subgroups of $\mathbb{R}^2$ under addition. The below demonstrates the checking of the Subgroup Criterion for H_3; the workings for H_1 are similar.

   SG1: $(0, 0) \in H_3$ as $3.0 + 0 = 0$, so $H_3 \neq \emptyset$.

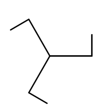
   SG2: Let $(a, b), (c, d) \in H_3$. Then $3b + a = 0$ and $3d + c = 0$. We want to show that $(a + c, b + d) \in H_3$. Indeed, $3(b + d) + (a + c) = (3b + a) + (3d + c) = 0 + 0 = 0$, so $(a + c, b + d) \in H_3$, as required.

   SG3: Let $(a, b) \in H_3$. Then $3b + a = 0$. We want to show that $(-a, -b) \in H_3$. Indeed, $3(-b) + (-a) = -(3b + a) = 0$, so $(-a, -b) \in H_3$, as required.

   Hence, by the Subgroup Criterion, H_3 is a subgroup of $\mathbb{R}^2$.

   Since $(0, -\frac{1}{3}) \in H_2$ but $(0, -\frac{1}{3}) + (0, -\frac{1}{3}) = (0, -\frac{2}{3}) \notin H_2$, it follows that SG2 fails and H_2 is not a subgroup of $\mathbb{R}^2$. Likewise, $(1, 1), (-1, 1) \in H_4$ but $(1, 1) + (-1, 1) = (0, 2) \notin H_4$, so H_4 is not a subgroup of $\mathbb{R}^2$.

3. Some possibilities are (a) a square; (b) a regular hexagon; (c) an equilateral triangle; (d) a capital letter 'F'; (e) a shape like

   .

4. In $\mathbb{Z}_3$, $\overline{2}^2 = \overline{1}$ so $\overline{2}$ has order 2. In $\mathbb{Z}_{11}$, $\overline{5}^2 = \overline{3}$, $\overline{5}^3 = \overline{4}$, $\overline{5}^4 = \overline{9}$, $\overline{5}^5 = \overline{1}$ so $\overline{5}$ has order 5. In $\mathbb{Z}_p$, $\overline{p-1}^2 = \overline{p^2 - 2p + 1} = \overline{1}$, so $\overline{p-1}$ has order 2.

5. (a) $\overline{5}$ has order 6 in $\mathbb{Z}_7 \setminus \{\overline{0}\}$. We want the value of $\overline{5^{3602}} = \overline{5}^{3602}$ in $\mathbb{Z}_7 \setminus \{0\}$. But, $\overline{5}^{6k} = \overline{1}$ for any integer k, so $\overline{5}^{3602} = \overline{5}^{3600}.\overline{5}^2 = \overline{5}^2 = \overline{4}$. Hence 5^{3602} days from today will be a Saturday.

   (b) Similar workings show $\overline{13}^2 = \overline{1}$ in $\mathbb{Z}_{24}$. Thus $\overline{13}^{123456789} = \overline{13}^{123456788}.\overline{13}^1 = \overline{13}$ so, $13^{123456789}$ hours from now adds 13 hours to the time. For example, if it is now 14:30GMT, then after $13^{123456789}$ hours it will be 03:30GMT.

For more details, start a thread on the discussion board.

Theorem (The Subgroup Criterion)**.** *Let G be a group and H be a subset of G. Then H is a subgroup of G if and only if the following three conditions hold.*

SG1: $H \neq \emptyset$.

SG2: $gh \in H$ *for all* $g, h \in H$.

SG3: $h^{-1} \in H$ *for all* $h \in H$.

For a group in additive notation, SG2 and SG3 become

SG2*:* $a + b \in H$ *for all* $a, b \in H$ *and* ***SG3****:* $-a \in H$ *for all* $a \in H$.