SURJECTIVITY, INJECTIVITY AND BIJECTIVITY

5 minute review. Recap the definitions of surjectivity, injectivity and bijectivity, drawing pictures if appropriate.

Class warm-up. Work through a few parts from Question 1 below (e.g. (a)-(c)).

Problems. Choose from the below.

1. For each function below, is it (i) surjective, (ii) injective, (ii) bijective? If you have answered no to (i) or (ii), give a clear counter-example to back up your claim. If you have answered yes to (iii), state the inverse function.
   (a) $f : \mathbb{R} \to \mathbb{R}, \quad f(a) = \sin a;$
   (b) $f : \mathbb{R} \to [-1, 1], \quad f(a) = \sin a;$
   (c) $f : [-\pi/2, \pi/2] \to [-1, 1], \quad f(a) = \sin a;$
   (d) $f : \mathbb{R} \to \mathbb{R}, \quad f(a) = a^4;$
   (e) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad f(a) = a^4;$
   (f) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad f(a) = a^4;$
   (g) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad f(a) = a^4;$
   (h) $f : \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}, \quad f(a) = a^2;$
   (i) $f : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash\{0\}, \quad f(a) = \frac{1}{a};$
   (j) $f : \mathbb{C} \to \mathbb{R}, \quad f(a) = |a|;$
   (k) $f : \mathbb{C} \to \mathbb{C}, \quad f(a) = \overline{a}.$
2. Let $f : \mathbb{Q} \to \mathbb{Q}$ be given by $f(a) = \dfrac{a^3 + 1}{a^3 - 2}$. Show that f is injective. Is f surjective?
3. Let $f : \mathbb{R}\backslash\{2\} \to \mathbb{R}$ be given by $f(a) = \dfrac{a - 3}{a - 2}$. Show that f is injective. Can you find $a \in \mathbb{R}\backslash\{2\}$ such that $f(a) = 1$? Is f bijective? If not, can you modify the codomain to make a bijective function (and prove that it is bijective)?
4. (a) Let $f : (1, \infty) \to \mathbb{R}$ be given by $f(x) = \dfrac{x + 1}{x^2 + 1}$. Show that f is injective.
   (b) What is the smallest value of a for which we can conclude that the function $f : (a, \infty) \to \mathbb{R}$ with the same rule as above is injective?

Homework: Chapter 1, Q9 from the problem booklet

Selected answers and hints.

1. Counter-examples to surjectivity and injectivity are recorded in the penultimate and final columns respectively.

Domain A	Codomain B	$f(a)$	Surj?	Inj?	Bij?	Element not hit	$a_1 \neq a_2$ but $f(a_1) = f(a_2)$
$\mathbb{R}$	$\mathbb{R}$	$\sin(a)$	no	no	no	2	$0, \pi$
$\mathbb{R}$	$[-1,1]$	$\sin(a)$	yes	no	no	none	$0, \pi$
$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1,1]$	$\sin(a)$	yes	yes	yes	none	none
$\mathbb{R}$	$\mathbb{R}$	a^4	no	no	no	-1	$-1, 1$
$\mathbb{R}_{\geq 0}$	$\mathbb{R}$	a^4	no	yes	no	-1	none
$\mathbb{R}$	$\mathbb{R}_{\geq 0}$	a^4	yes	no	no	none	$-1, 1$
$\mathbb{R}_{\geq 0}$	$\mathbb{R}_{\geq 0}$	a^4	yes	yes	yes	none	none
$\mathbb{Q}_{\geq 0}$	$\mathbb{Q}_{\geq 0}$	a^2	no	yes	no	2	none
$\mathbb{Q}\backslash\{0\}$	$\mathbb{Q}\backslash\{0\}$	$\frac{1}{a}$	yes	yes	yes	none	none
$\mathbb{C}$	$\mathbb{R}$	$\lvert a \rvert$	no	no	no	-1	$1, i$
$\mathbb{C}$	$\mathbb{C}$	$\overline{a}$	yes	yes	yes	none	none

The inverses asked for are (c) $g : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $g(a) = \sin^{-1}(a)$; (g) $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $g(a) = a^{\frac{1}{4}}$; (i) $g : \mathbb{Q}\backslash\{0\} \to \mathbb{Q}\backslash\{0\}$, $g(a) = \frac{1}{a}$; (k) $g : \mathbb{C} \to \mathbb{C}$, $g(a) = \overline{a}$.

2. Let $a_1, a_2 \in \mathbb{Q}$ be such that $f(a_1) = f(a_2)$. Then $\frac{a_1^3+1}{a_1^3-2} = \frac{a_2^3+1}{a_2^3-2}$. Hence $(a_2^3-2)(a_1^3+1) = (a_1^3-2)(a_2^3+1)$, so $a_2^3a_1^3 + a_2^3 - 2a_1^3 - 2 = a_1^3a_2^3 + a_1^3 - 2a_2^3 - 2$. Hence $a_1^3 = a_2^3$ and therefore $a_1 = a_2$. Thus f is injective.

   We find that f is not surjective, because there is no $a \in \mathbb{Q}$ with $f(a) = 1$. (If $f(a) = 1$ then $a^3 + 1 = a^3 - 2$ which is impossible).

3. Let $a_1, a_2 \in \mathbb{R}\backslash\{2\}$ be such that $f(a_1) = f(a_2)$. Then $\frac{a_1-3}{a_1-2} = \frac{a_2-3}{a_2-2}$. Hence $(a_2-2)(a_1-3) = (a_1-2)(a_2-3)$, so $a_2a_1 - 2a_1 - 3a_2 + 6 = a_1a_2 - 2a_2 - 3a_1 + 6$. Therefore $a_1 = a_2$. Thus f is injective.

   Suppose that there exists $a \in \mathbb{R}\backslash\{2\}$ such that $f(a) = 1$. Then $\frac{a-3}{a-2} = 1$. Hence $a - 2 = a - 3$, which is impossible, so no such a exists. Hence f is not surjective, so not bijective.

   The function $f : \mathbb{R}\backslash\{2\} \to \mathbb{R}\backslash\{1\}$ given by the same rule is bijective. To see this, the same argument as above shows that f is injective. To show that f is surjective, let $b \in \mathbb{R}\backslash\{1\}$ and let $a = \frac{2b-3}{b-1}$ (obtained by setting $f(a) = b$ and rearranging to make a the subject). Note that $a \neq 2$, as otherwise $2b - 3 = 2b - 2$, which is impossible, so $a \in \mathbb{R}\backslash\{2\}$. Now

$$f(a) = f\left(\frac{2b-3}{b-1}\right) = \frac{\left(\frac{2b-3}{b-1}\right) - 3}{\left(\frac{2b-3}{b-1}\right) - 2} = \frac{(2b-3) - 3(b-1)}{(2b-3) - 2(b-1)} = b.$$

   Thus f is surjective.

4. (a) Suppose that $a_1, a_2 \in (1, \infty)$ such that $f(a_1) = f(a_2)$. Then $\frac{a_1+1}{a_1^2+1} = \frac{a_2+1}{a_2^2+1}$. Rearranging, we get $(a_1^2a_2 - a_1a_2^2) + (a_1^2 - a_2^2) - (a_1 - a_2) = 0$. Each bracket on the left-hand side has a factor of $(a_1 - a_2)$, and we arrive at $(a_1 - a_2)(a_1a_2 + a_1 + a_2 - 1) = 0$. As $a_1, a_2 > 1$, it follows that $a_1a_2 + a_1 + a_2 - 1 > 0$ so we must have $a_1 = a_2$ and f is injective.

   (b) If $a_1, a_2 > \sqrt{2} - 1$, then $a_1a_2 + a_1 + a_2 - 1 > 0$ so, as above, f is injective on $(\sqrt{2} - 1, \infty)$. As there's a local maximum at $x = \sqrt{2} - 1$, it follows that we lose injectivity if we try to extend the domain further.