

FUNCTIONS

5 minute review. Recap the composition $g \circ f$ for functions $f : A \rightarrow B$ and $g : B \rightarrow C$, what it means for two functions $f, g : A \rightarrow A$ to commute and what it means for a function to have an inverse. (Surjectivity/injectivity will be covered in this module later, although it has been seen in MAS110 and MAS114 Semester 1.)

Class warm-up. Let $A = \mathbb{R}$ and $B = [0, \infty)$. Let $f : A \rightarrow B$ be given by $f(x) = x^2$ and $g : B \rightarrow A$ be given by $g(y) = \sqrt{y}$.

- (a) Is $f \circ g = \text{id}_B$? (b) Is $g \circ f = \text{id}_A$? (c) Is g an inverse of f ?

Problems. Choose from the below.

- For fixed $a, b, c, d \in \mathbb{R}$, let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = ax + b$ and $g(x) = cx + d$.
 - Let $a = 4, b = 2, c = 7$. Given that f and g commute, find d .
 - Now let $a = b = 1$ and let $c = 2$. Show that f and g do not commute for any value of d .
- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \sin(x)$. The inverse trigonometric function \sin^{-1} or arcsin has domain $[-1, 1]$ and codomain $[-\pi/2, \pi/2]$.
 - Give an example of a real number x such that $\sin^{-1}(\sin x) \neq x$. Is \sin^{-1} an inverse for f ?
 - Identify subsets A and B of \mathbb{R} such that \sin^{-1} is an inverse for the function $g : A \rightarrow B$ given by $g(a) = \sin(a)$.
- Let $\alpha, \beta \in \mathbb{R}$. By showing that they have the same effect on a general point $(r, \theta) \in \mathbb{R}^2$, show that $\text{rot}_\alpha \text{ref}_\beta = \text{ref}_{\alpha+\beta}$ and $\text{ref}_\alpha \text{rot}_\beta = \text{ref}_{\alpha-\beta}$ as functions $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.
 - In each of the following, find a rotation or reflection f which satisfies the given equation.
 - $f \circ \text{ref}_{\frac{2\pi}{3}} = \text{rot}_{\frac{\pi}{2}}$; (ii) $f \circ \text{rot}_{\frac{2\pi}{3}} = \text{ref}_{\frac{\pi}{6}}$; (iii) $f \circ \text{ref}_{\frac{2\pi}{3}} = \text{ref}_{\frac{\pi}{2}}$.
 - Show that ref_α commutes with the rotations rot_0 and rot_π , and find two reflections which commute with ref_α .
- Let $\alpha, \beta \in \mathbb{R}$. Show that $\text{ref}_\alpha \text{rot}_\beta = \text{rot}_\beta \text{ref}_\alpha$ if and only if $\text{rot}_\beta = \text{rot}_0$ or $\text{rot}_\beta = \text{rot}_\pi$. (Hint: you may have already proved one direction of this.)
 - Let $\alpha, \beta \in \mathbb{R}$. Show that $\text{ref}_\alpha \text{ref}_\beta = \text{ref}_\beta \text{ref}_\alpha$ if and only if $\text{ref}_\beta = \text{ref}_\alpha$ or $\text{ref}_\beta = \text{ref}_{\pi+\alpha}$.

Homework: Chapter 1, Q5 from the problem booklet

For the warm-up, (a) yes: if $y \geq 0$, then $f \circ g(y) = (\sqrt{y})^2 = y$; (b) no: as a counter-example, $g \circ f(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$; (c) no, because of (b). Restricting the domain of f and the codomain of g to be B will make g into an inverse for f .

Selected answers and hints.

1. (a) Here $f \circ g(x) = f(7x + d) = 28x + 4d + 2$, whereas $g \circ f(x) = g(4x + 2) = 28x + 14 + d$. Therefore f and g commute precisely when $4d + 2 = 14 + d$, that is when $d = 4$.

(b) Here $f \circ g(x) = f(2x + d) = 2x + d + 1$, whereas $g \circ f(x) = g(x + 1) = 2x + 2 + d$. Therefore f and g commute precisely when $d + 1 = 2 + d$, that is for no value of d .
2. (a) Take $x = \pi$. Then $\sin \pi = 0$ and $\sin^{-1}(0) = 0$ so $\sin^{-1}(\sin \pi) = 0 \neq \pi$. (Any value outside $[-\pi/2, \pi/2]$ will do.)

(b) $A = [-\pi/2, \pi/2]$ and $B = [-1, 1]$.
3. (a) Let $(r, \theta) \in \mathbb{R}^2$. Then $\text{rot}_\alpha \text{ref}_\beta(r, \theta) = \text{rot}_\alpha(\text{ref}_\beta(r, \theta)) = \text{rot}_\alpha(r, \beta - \theta) = (r, \alpha + (\beta - \theta)) = (r, (\alpha + \beta) - \theta) = \text{ref}_{\alpha+\beta}(r, \theta)$. Hence $\text{rot}_\alpha \text{ref}_\beta = \text{ref}_{\alpha+\beta}$.

Let $(r, \theta) \in \mathbb{R}^2$. Then $\text{ref}_\alpha \text{rot}_\beta(r, \theta) = \text{ref}_\alpha(\text{rot}_\beta(r, \theta)) = \text{ref}_\alpha(r, \beta + \theta) = (r, \alpha - (\beta + \theta)) = (r, (\alpha - \beta) - \theta) = \text{ref}_{\alpha-\beta}(r, \theta)$. Hence $\text{ref}_\alpha \text{rot}_\beta = \text{ref}_{\alpha-\beta}$.

(b) (i) $f = \text{ref}_{\frac{7\pi}{6}}$; (ii) $f = \text{ref}_{\frac{5\pi}{6}}$; (iii) $f = \text{rot}_{-\frac{\pi}{6}} = \text{rot}_{\frac{11\pi}{6}}$.

(c) We have $\text{ref}_\alpha \text{rot}_0 = \text{ref}_\alpha = \text{rot}_0 \text{ref}_\alpha$, so ref_α commutes with rot_0 . Similarly, $\text{ref}_\alpha \text{rot}_\pi = \text{ref}_{\alpha-\pi} = \text{ref}_{\alpha-\pi+2\pi} = \text{ref}_{\alpha+\pi} = \text{rot}_\pi \text{ref}_\alpha$, so ref_α commutes with rot_π .

We are looking for two values of β for which $\text{ref}_\alpha \text{ref}_\beta = \text{ref}_\beta \text{ref}_\alpha$, that is $\text{rot}_{\alpha-\beta} = \text{rot}_{\beta-\alpha}$ (using the rot/ref formulae). The most obvious choice is when $\beta = \alpha$, but $\beta = \alpha + \pi$ also works, as $\text{rot}_{\alpha-(\alpha+\pi)} = \text{rot}_{-\pi} = \text{rot}_\pi = \text{rot}_{(\alpha+\pi)-\alpha}$. So two reflections that work are ref_α and $\text{ref}_{\alpha+\pi}$. (Note that these are distinct reflections; that is, $\text{ref}_\alpha \neq \text{ref}_{\alpha+\pi}$.)

4. (a) In 3(c) above we've already shown that both rot_0 and rot_π commute with ref_α ; that is, if $\text{rot}_\beta = \text{rot}_0$ or $\text{rot}_\beta = \text{rot}_\pi$ then $\text{ref}_\alpha \text{rot}_\beta = \text{rot}_\beta \text{ref}_\alpha$.

For the converse, suppose that $\text{ref}_\alpha \text{rot}_\beta = \text{rot}_\beta \text{ref}_\alpha$. Then, by the rot/ref formulae, $\text{ref}_{\alpha-\beta} = \text{ref}_{\beta+\alpha}$. Hence $\beta + \alpha = \alpha - \beta + 2n\pi$ for some integer n .¹ Therefore $\beta = n\pi$ for some integer n . If n is even then $\text{rot}_\beta = \text{rot}_0$ and if n is odd then $\text{rot}_\beta = \text{rot}_\pi$. Thus either $\text{rot}_\beta = \text{rot}_0$ or $\text{rot}_\beta = \text{rot}_\pi$.

- (b) In 3(c) we've shown that both ref_α and $\text{ref}_{\alpha+\pi}$ commute with ref_α ; that is, if $\text{ref}_\beta = \text{ref}_\alpha$ or $\text{ref}_\beta = \text{ref}_{\alpha+\pi}$ then $\text{ref}_\alpha \text{ref}_\beta = \text{ref}_\beta \text{ref}_\alpha$.

For the converse, suppose that $\text{ref}_\alpha \text{ref}_\beta = \text{ref}_\beta \text{ref}_\alpha$. Then, by the rot/ref formulae, $\text{rot}_{\alpha-\beta} = \text{rot}_{\beta-\alpha}$. Hence $\beta - \alpha = \alpha - \beta + 2n\pi$ for some integer n . Therefore $\beta = \alpha + n\pi$ for some integer n . If n is even then $\text{ref}_\beta = \text{ref}_\alpha$ and if n is odd then $\text{ref}_\beta = \text{ref}_{\alpha+\pi}$. Thus either $\text{ref}_\beta = \text{ref}_\alpha$ or $\text{ref}_\beta = \text{ref}_{\alpha+\pi}$.

For more details, start a thread on the discussion board.

¹Important! Remember that we can't conclude here that $\beta + \alpha = \alpha - \beta$, but instead that they must differ by a multiple of 2π .