The Morava E-theories of finite general linear groups

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Abstract

By studying the representation theory of a certain infinite p-group and using the generalised characters of Hopkins, Kuhn and Ravenel we find useful ways of understanding the rational Morava E-theory of the classifying spaces of general linear groups over finite fields. Making use of the well understood theory of formal group laws we establish more subtle results integrally, building on relevant work of Tanabe. In particular, we study in detail the cases where the group has dimension less than or equal to the prime p at which the E-theory is localised.

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Chapter 1

Introduction

1.1 Project overview

1.1.1 Background

For any finite group G there is an associated topological space, BG, known as the classifying space for G. This space is closely related in structure to the group itself and lends itself well to being studied by cohomological methods (see [Ben91]). We will look at groups of the form $GL_d(K)$, where K is a finite field, and their associated classifying spaces.

We study the spaces $BGL_d(K)$ using a family of generalised cohomology theories known as the Morava E-theories. For each integer $n \geq 1$ and each prime p there is an even-periodic cohomology theory E with coefficient ring $E^* = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}]$, where $\mathbb{Z}_p$ denotes the p-adic integers, $u_1, \ldots, u_{n-1}$ all lie in degree zero and u is an invertible element in degree -2. These theories turn out to be computable yet, taken together, give a great deal of information (see [Rav92]). As covered in Chapter 4, the Morava E-theories are complex oriented and have close relations with the theory of formal group laws. Under mild hypotheses, reduction modulo the ideal $(p, u_1, \ldots, u_{n-1})$ gives a related theory, K, with which is associated the theory of formal group laws over finite fields.

The starting point for the calculation of $E^*(BGL_d(K))$ is the work of Friedlander and Mislin ([FM84], [Fri82]) who showed that, whenever l is a prime different to p, the mod p cohomology of $BGL_d(\overline{\mathbb{F}}_l)$ coincides with that of $BGL_d(\mathbb{C})$, where $\overline{\mathbb{F}}_l$ denotes an algebraic closure of the finite field with l elements. Borel ([Bor53]) had already shown that, letting T denote the maximal torus in $GL_d(\mathbb{C})$, the latter could be described in terms of the invariant elements of the cohomology of BT under the permutation action of the relevant symmetric group.

In [Tan95], Tanabe used the above ideas to show that, for a theory $K(n)$ closely related to K above, the $K(n)$-cohomology of $BGL_d(\mathbb{F}_{l^r})$ can be recovered from that of $BGL_d(\overline{\mathbb{F}}_l)$ as the coinvariants under the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_{l^r})$, and that $K(n)^*(BGL_d(\overline{\mathbb{F}}_l))$ is just a power series ring over $\mathbb{F}_p$.

There are some general techniques due to Hovey and Strickland ([HS99]) that enable results from the theory $K(n)$ to be carried over into E-theory. In this vein, an extensive study of the E-theory of $B\Sigma_d$ has been carried out by the latter author in [Str98]. Also relevant is [Str00] where it was shown that, for finite G, $E^*(BG)$ has duality over its coefficient ring.

The most useful tool for our understanding turns out to be the generalised character theory of Hopkins, Kuhn and Ravenel ([HKR00]). There they show that, rationally, studying the E-theory of BG for a finite group G reduces to understanding commuting n-tuples of p-elements of G. We find that when $G = GL_d(K)$ for some finite field K of characteristic not equal to p this reduces to understanding the K-representation theory of the group $\mathbb{Z}_p^n$.

1.1.2 Thesis outline

In Chapter 2 we outline the basic material and conventions used in the thesis looking, in particular, at finite fields, local rings and the p-adic integers. We also explore the notion of duality algebras and make some preliminary calculations, establishing some basic results on the p-divisibility of integers of the form $k^s - 1$.

Fixing a prime p, in Chapter 3 we study the p-local structure of the finite general linear groups $GL_d(K)$ for finite fields K of characteristic different from p and find that it relates closely to that of the symmetric group Σ_d. We give particular attention to the groups of dimension p over fields for which p divides $|K^\times|$ and find that they have only two different conjugacy classes of abelian p-subgroups, one being represented by the maximal torus and the other by a cyclic group. We also look at the normalizer of the maximal torus in this case, finding that it has a finer p-local structure.

Chapter 4 details the relevant theory of formal group laws and defines the standard p-typical formal group law that is fundamental to the development of the Morava E-theories, which we later go on to define. We also outline the relevant known results in the Morava E-theory of classifying spaces, using the relationship between mod p cohomology and the Morava K- and E-theories. We show that, for all of the groups G we consider, the Morava E-theory of BG is free and lies in even degrees. It follows that the Morava K-theory of BG is recoverable from the E-theory in simple algebraic terms. We introduce variants of the standard chern and euler classes which prove to be more convenient in our setting.

In Chapter 5 we look at the generalised character theory of Hopkins, Kuhn and Ravenel and apply it to the general linear group $GL_d(\mathbb{F}_q)$, where $q = l^r$ is a power of a prime different to p. We introduce the groups $\Theta^* = \mathbb{Z}_p^n$, $\Phi = (\mathbb{Z}/p^\infty)^n$ (where $\mathbb{Z}/p^\infty = \varinjlim (\mathbb{Z}/p^k)$) and $\Lambda = \langle q \rangle \leqslant \mathbb{Z}_p^\times$. We let Λ act on Φ and find that the set of d-dimensional $\mathbb{F}_q$-representations of Θ^* bijects with $(\Phi^d/\Sigma_d)^\Lambda$ (see Theorem A). We also give thought to the cases where d is less than or equal to p finding that, under the hypothesis that p divides $q-1$, we can understand the latter set well. The generalised character theory then gives us a complete description of $L \otimes_{E^*} E^*(BGL_d(\mathbb{F}_q))$, where L is some extension of $\mathbb{Q} \otimes E^*$.

The aim in Chapter 6 is to get a good description of $E^*(BGL_d(\mathbb{F}_q))$ for the cases where d is at most p and p divides $q-1$. We show that Tanabe's results on the Morava K-theory of the relevant spaces lifts to E-theory; that is, we have $E^*(BGL_d(\mathbb{F}_q)) \simeq E^*(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ where $\Gamma = \mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_q)$ acts on $GL_d(\overline{\mathbb{F}}_l)$ component-wise and hence also on its cohomology. Letting $T_d \simeq (\mathbb{F}_q^\times)^d$ denote the maximal torus of $GL_d(\mathbb{F}_q)$ and letting Σ_d act by permuting the coordinates we show that the restriction map $\beta : E^*(BGL_d(\mathbb{F}_q)) \to E^*(BT)$ has image $E^*(BT)^{\Sigma_d}$. Further, when $d < p$ this map is an isomorphism onto its image (Theorem B). For the case $d = p$, we choose a basis for $\mathbb{F}_{q^p}$ over $\mathbb{F}_q$ to get an embedding $\mathbb{F}_{q^p}^\times \rightarrowtail GL_p(\mathbb{F}_q)$ and hence a map in E-theory $E^*(BGL_p(\mathbb{F}_q)) \to E^*(B\mathbb{F}_{q^p}^\times)$. There is a quotient ring D of $E^*(B\mathbb{F}_{q^p}^\times)$ and we let α be the composition $E^*(BGL_p(\mathbb{F}_q)) \to D$. Since $\Gamma = \mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_q)$ acts on $\mathbb{F}_{q^p}$ it also acts on D and we show that α has image D^Γ. We find that α and β are jointly injective and that they induce an isomorphism $\mathbb{Q} \otimes E^*(BGL_p(\mathbb{F}_q)) \simeq \mathbb{Q} \otimes E^*(BT)^{\Sigma_p} \times \mathbb{Q} \otimes D^\Gamma$.

We also show that, in this situation, the kernel of β is principal and we are able to give an explicit basis for $E^*(BGL_p(\mathbb{F}_q))$ over E^* (see Theorem C).

1.2 Acknowledgements

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1.3 Notational conventions

- For a positive integer k we define $C_k = \{z \in S^1 \mid z^k = 1\}$ to be the cyclic subgroup of S^1 of order k.
- Let H be a subgroup of G. Then we write $N_G(H)$ for the normalizer of H in G.
- Given a group G and a prime p we will write $\mathrm{Syl}_p(G)$ to denote a Sylow p-subgroup of G. Note that $\mathrm{Syl}_p(G)$ is determined up to non-canonical isomorphism.
- For a group G and an element $g \in G$ we write conj_g for the conjugation map $G \to G$, $h \mapsto ghg^{-1}$.
- Given a group G acting on a set S we let S^G denote the G-invariant elements of S. Similarly, if G acts on a ring R we let R_G denote the coinvariants of the action; that is, $R_G = R/(r - g.r \mid r \in R, g \in G)$.
- Given a ring R we denote by $R[x]$ the ring of polynomials in x with coefficients in R and, likewise, by $R[\![x]\!]$ the ring of formal power-series.
- We will write $R^\times$ for the group of units of a ring R under multiplication. If $a, b \in R$ then we write $a \sim b$ to denote that a is a unit multiple of b in R.
- We write $\mathrm{nil}(R)$ for the nilradical and $\mathrm{rad}(R)$ for the Jacobson radical of a ring R, the former being the set of nilpotents and the latter the intersection of the maximal ideals.
- Unless otherwise indicated, the symbol $\otimes$ will denote the tensor product over $\mathbb{Z}$.
- We will write $\mathrm{Hom}(-,-)$ for the set of homomorphisms between two objects, where the structure should be clear from the context, and $\mathrm{Aut}(-)$ for the set of automorphisms of an object. We will write $\mathrm{Map}(-,-)$ for the set of functions between two sets.
- We will use H to denote singular homology and cohomology.
- For a (generalised) cohomology theory h we will write $h^* = h^*(pt)$ to denote the ring of coefficients.
- Given a subspace $Y \subseteq X$ we write res^X_Y for the map in cohomology $h^*(X) \to h^*(Y)$.

Chapter 2

Preliminaries

2.1 Definitions, conventions and preliminary results

We outline some of the basic definitions and results needed. Unless otherwise stated, all rings and algebras are commutative and unital with homomorphisms respecting the units. All of the material presented here is well known and good reference texts include [Lan02], [Ben91] and [Mat89].

2.1.1 Local rings

A ring R is known as a *local ring* if it has precisely one maximal ideal. We write $(R, \mathfrak{m})$ to denote the local ring R with maximal ideal $\mathfrak{m}$. It is easy to show that R is local if and only if $R \setminus R^\times$ is an ideal in R, since every element of a proper ideal is a non-unit and every non-unit generates a proper ideal. This ideal will necessarily be the unique maximal ideal of R. If I is any ideal in R then the ring R/I is again local with maximal ideal $\mathfrak{m}/I$. We will need the following result.

Lemma 2.1. *If $(R, \mathfrak{m})$ is a local ring then so is $R[\![x]\!]$ with maximal ideal $\mathfrak{m}[\![x]\!] + (x)$.*

Proof. Any power series $f(x) \in R[\![x]\!]$ is invertible if and only if $f(0) \in R^\times$ (see, for example, [Frö68, Proposition 1]). Since $\mathfrak{m} = R \setminus R^\times$ it follows that $R[\![x]\!] \setminus (R[\![x]\!])^\times$ is the ideal $\mathfrak{m}[\![x]\!] + (x)$. □

Given any ring R and an ideal I in R we define the *I-adic topology* on R to be the topology generated by the open sets $x + I^n$ ($x \in R$, $n \in \mathbb{N}$). If $\bigcap_n I^n = 0$ then this topology coincides with the one given by the metric

$$d(a,b) = \begin{cases} 2^{-n} & \text{if } a - b \in I^n \text{ but } a - b \notin I^{n+1} \\ 0 & \text{if } a - b \in I^n \text{ for all } n \end{cases}$$

where we use the convention that $I^0 = R$. Here the number 2 occurring is arbitrary; we get an equivalent metric choosing any real number greater than 1.

If not otherwise stated we assume that a local ring $(R, \mathfrak{m})$ carries the $\mathfrak{m}$-adic topology. In particular, by a *complete local ring* we will mean a local ring that it is complete with respect to the topology generated by its maximal ideal.

If $(R, \mathfrak{m})$ is a local ring we define the *socle* of R, denoted $\mathrm{soc}(R)$, to be the annihilator of the maximal ideal $\mathfrak{m}$. That is, $\mathrm{soc}(R) = \mathrm{ann}_R(\mathfrak{m}) = \{r \in R \mid r\mathfrak{m} = 0\}$. Since $\mathfrak{m}.\mathrm{soc}(R) = 0$ it follows that $\mathrm{soc}(R)$ is a vector space over the field $R/\mathfrak{m}$.

2.1.2 The p-adic numbers

Let p be a prime. We define the *p-adic integers*, denoted $\mathbb{Z}_p$, to be the completion of $\mathbb{Z}$ with respect to the (p)-adic topology. Note that, since $\bigcap_n (p^n) = 0$, $\mathbb{Z}_p$ is in fact a metric space. The ideal $p\mathbb{Z}_p$ is the unique maximal ideal of $\mathbb{Z}_p$ and hence $\mathbb{Z}_p$ is a complete local ring.

An alternative characterisation of the p-adic integers is obtained by considering the inverse system

$$\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \dots$$

and defining $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$. In this way, it is clear that $\mathbb{Z}_p$ carries the structure of a commutative ring and we topologise it using the norm $|a|_p = p^{-v_p(a)}$, where v_p is the *p-adic valuation* given by

$$v_p(a) = \begin{cases} n & \text{if } a = 0 \text{ in } \mathbb{Z}/p^n \text{ but } a \neq 0 \text{ in } \mathbb{Z}/p^{n+1} \\ \infty & \text{if } a = 0. \end{cases}$$

With the latter definition in mind, it is perhaps easiest to think of $\mathbb{Z}_p$ as the set of sequences of integers $(a_0, a_1, \dots)$ such that $a_{k+1} = a_k \bmod p^k$ with componentwise multiplication and addition. Another useful observation is that every p-adic integer a can be given a unique expansion $a = \sum_{i=0}^{\infty} a_i p^i$ where $a_i \in \{0, \dots, p-1\}$ for each i.

We can equally well apply the norm $|-|_p$ to $\mathbb{Q}$, defining $v_p(p^n \frac{s}{t}) = n$ whenever both s and t are coprime to p. The completion of $\mathbb{Q}$ with respect to $|-|_p$ then gives the field of p-adic numbers, denoted $\mathbb{Q}_p$. There is a presentation

$$\mathbb{Q}_p = \left\{ \frac{a}{p^n} \mid a \in \mathbb{Z}_p \text{ and } n \geq 0 \right\},$$

and, similarly to above, every p-adic number b has a unique expansion $b = \sum_k^{\infty} b_i p^i$ for some $k \leq 0$, where $b_i \in \{0, \dots, p-1\}$ for each $i \geq k$. Note that $\mathbb{Z}_p$ is a subring, and hence an additive subgroup, of $\mathbb{Q}_p$.

A related construction is that of the *Prüfer group*. There is a direct system of embeddings

$$\mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p^3 \rightarrow \dots$$

with each map corresponding to multiplication by p and we define

$$\mathbb{Z}/p^\infty = \varinjlim \mathbb{Z}/p^n = \bigcup_n \mathbb{Z}/p^n.$$

There is a canonical identification $\mathbb{Z}/p^\infty \simeq \{z \in S^1 \mid z^{p^n} = 1 \text{ for some } n\}$, although $\mathbb{Z}/p^\infty$ usually carries the discrete topology rather than that of the subspace of S^1.

Lemma 2.2. *There are isomorphisms Hom$(\mathbb{Z}/p^\infty, S^1) \simeq \mathbb{Z}_p$ and $\mathbb{Z}/p^\infty \simeq \mathbb{Q}_p/\mathbb{Z}_p$.*

Proof. View $\mathbb{Z}/p^\infty$ as a subgroup of S^1 and let $\phi : \mathbb{Z}/p^\infty \rightarrow S^1$ be any homomorphism. Then, for each $n \geq 0$, $\phi(e^{2\pi i/p^n}) = e^{2k_n \pi i/p^n}$ for some $k_n \in \mathbb{Z}/p^n$. Since $k_{n+1} = k_n \bmod p^n$ for each n we have a well defined element $(k_n) \in \mathbb{Z}_p$. Conversely, any $a \in \mathbb{Z}_p$ gives a unique

$\phi_a \in \mathrm{Hom}(\mathbb{Z}/p^\infty, S^1)$ defined by $\phi_a(e^{2\pi i/p^n}) = e^{2a_n\pi i/p^n}$. It is clear that this construction gives an isomorphism of the groups.

For the second statement, the homomorphism $\mathbb{Z}/p^\infty \to \mathbb{Q}_p/\mathbb{Z}_p$ given by $e^{2k\pi i/p^n} \mapsto k/p^n + \mathbb{Z}_p$ is well defined and injective. It is surjective since, for $a \in \mathbb{Z}_p$, we have $a/p^n + \mathbb{Z}_p = \left(\sum_{i=0}^{n-1} a_i p^i\right)/p^n + \mathbb{Z}_p$ with the latter visibly lying in the image. □

2.1.3 The Teichmüller lift map

The ring of p-adic integers, $\mathbb{Z}_p$, contains precisely $p-1$ roots of $x^{p-1}-1=0$. Further, these are all distinct (and necessarily non-zero) mod p. We define the *Teichmüller lift map* to be the monomorphism of groups $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ sending a to the unique $(p-1)^{\text{th}}$ root of unity congruent to a mod p. We often write $\hat{a}$ for $\omega(a) \in \mathbb{Z}_p^\times$. We will need the following result.

Lemma 2.3. *With the notation above we have* $\displaystyle\prod_{a\in(\mathbb{Z}/p)^\times} \hat{a} = -1 \in \mathbb{Z}_p^\times$.

Proof. This is an immediate corollary to the analogous result for $(\mathbb{Z}/p)^\times$; we briefly outline the details. Suppose $a \in \mathbb{Z}/p$ with $a^2 = 1$. Then $a^2 - 1 = 0$ so that $(a+1)(a-1) = 0$. Hence, since $\mathbb{Z}/p$ is a field, we have $a = \pm 1$, both of which are indeed square roots of 1. Thus, the only self-inverse elements in $\mathbb{Z}/p$ are ± 1. Hence

$$\prod_{a\in(\mathbb{Z}/p)^\times} a = 1 \times (-1) \times \prod_{\substack{a\in(\mathbb{Z}/p)^\times \\ a\neq\pm 1}} a = -1$$

since the latter product comprises pairs of inverse elements. Applying the homomorphism ω then gives the result. □

2.1.4 Finite fields and their algebraic closures

For each prime p we define $\mathbb{F}_p$ to be the field $\mathbb{Z}/p$. It is well known that we can choose an algebraic closure $\overline{\mathbb{F}}_p$ for $\mathbb{F}_p$ and from here on we will assume that we have done so for each prime. For a natural number r we then define

$$\mathbb{F}_{p^r} = \left\{ a \in \overline{\mathbb{F}}_p \mid a^{p^r} = 1 \right\}$$

which is a field containing p^r elements. The two definitions coincide for the case $r = 1$. It is a classical result that, for each r, $\mathbb{F}_{p^r}$ is the unique field containing p^r elements up to a non-canonical isomorphism (see, for example, [Lan02]). Further, every finite field is isomorphic to $\mathbb{F}_{p^r}$ for some p and r. We refer to p as the *characteristic* of the field. Note that the characteristic of a finite field K is given by $\mathrm{char}(K) = \min\{n \in \mathbb{N} \mid n.1 = 0 \text{ in } K\}$.

If K is a field of characteristic p and $\overline{K}$ is an algebraic closure for K then the map $F : \overline{K} \to \overline{K}$ sending $a \mapsto a^p$ satisfies $F(1) = 1$, $F(ab) = F(a)F(b)$ and

$$F(a+b) = (a+b)^p = a^p + b^p = F(a) + F(b)$$

since $p = 0$ in K and $p|\binom{p}{i}$ for $i \neq 0, p$. It follows that F is a homomorphism of fields, and we refer to F as the *Frobenius homomorphism*. For any prime p the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$

is isomorphic to the profinite integers (see [Wei94, p207]), topologically generated by F. We will write $\Gamma = \Gamma_p = \langle F \rangle \simeq \mathbb{Z}$ for the subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ generated by F.

Another important property of finite fields is that their multiplicative group of units is always cyclic (again, see [Lan02]) and, for each p, there is a (non-canonical) embedding $\overline{\mathbb{F}}_p^\times \to S^1$. If l and p are distinct primes then the embedding $\overline{\mathbb{F}}_l^\times \to S^1$ induces a group isomorphism

$$\{a \in \overline{\mathbb{F}}_l^\times \mid a^{p^n} = 1 \text{ for some } n\} \simeq \mathbb{Z}/p^\infty$$

where $\mathbb{Z}/p^\infty$ is the Prüfer group of Section 2.1.2. We will assume, from here on, that we have chosen such embeddings for each prime. Note, however, that $\overline{\mathbb{F}}_l$ and $\overline{\mathbb{F}}_l^\times$ still both carry the discrete topology.

2.1.5 The symmetric and general linear groups

For each $d \geq 1$ the *symmetric group on d symbols*, denoted Σ_d, is the group of permutations of the finite set $\{1, \ldots, d\}$. For any s and t there is an obvious embedding $\Sigma_s \times \Sigma_t \rightarrowtail \Sigma_{s+t}$. In particular, we can view Σ_{d-1} as the subgroup of Σ_d fixing d. We will refer to the permutation $(1 \ldots d) \in \Sigma_d$ as the *standard d-cycle* and denote it by γ_d.

Let K be a field. Then the *general linear group over K of dimension d* is the group of invertible $d \times d$ matrices with entries in K and is denoted $GL_d(K)$. Equivalently, it is the group of automorphisms of the d-dimensional vector space K^d. Similarly to above, there is an obvious embedding $GL_s(K) \times GL_t(K) \rightarrowtail GL_{s+t}(K)$ for any s and t. We can also view Σ_d as a subgroup of $GL_d(K)$ via the map $\sigma \mapsto (\sigma_{ij})$ where

$$\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

and our convention is that σ_{ij} denotes the entry in the i^{th} row and j^{th} column. Another important subgroup of $GL_d(K)$ is the embedding of $(K^\times)^d$ along the diagonal, which we refer to as the *maximal torus* and denote by T_d.

2.1.6 Semidirect and wreath products

Let G, H be groups and let G act on H via group automorphisms. Then we define the *semidirect product* of G and H, written $G \ltimes H$, to be the group with underlying set $G \times H$ but multiplication given by

$$(g_1; h_1).(g_2; h_2) = (g_1 g_2; (g_2^{-1}.h_1)h_2).$$

Note that there is an exact sequence $1 \to H \rightarrowtail G \ltimes H \twoheadrightarrow G \to 1$. One of the main sources of semidirect products is the following.

Proposition 2.4. *Let G and H be subgroups K with $G \cap H = 1$ and $G \leqslant N_K(H)$. Then G acts on H by $g.h = ghg^{-1}$ and GH is a subgroup of K isomorphic to $G \ltimes H$.*

Proof. Since G is contained in the normaliser of H in K the action of G on H is well defined and it is straightforward to check that GH is a subgroup of K. Now, define a map

$\phi : G \ltimes H \to GH$ by $\alpha(g;h) = gh$. Then ϕ is clearly surjective. To see that it is a group homomorphism, we have

$$\begin{aligned}
\phi((g_1;h_1)(g_2;h_2)) &= \phi(g_1g_2;(g_2^{-1}.h_1)h_2) \\
&= g_1g_2(g_2^{-1}h_1g_2)h_2 \\
&= g_1h_1g_2h_2 \\
&= \phi(g_1;h_1)\phi(g_2;h_2).
\end{aligned}$$

Finally, for injectivity note that if $gh = 1$ then $g = h^{-1} \in G \cap H = 1$ whereby $g = h = 1$. □

Now, let $S \leqslant \Sigma_d$ for some d and let G be any group. Then S acts on G^d by $\sigma.(g_1, \dots, g_d) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(d)})$ and we define the *wreath product* of S and G by $S \wr G = S \ltimes G^d$. One important property of the wreath product is that it is associative in the following sense.

Lemma 2.5. *Let $A \leqslant \Sigma_s$ and $B \leqslant \Sigma_t$ and G be any group. Then there is a canonical embedding $A \wr B \rightarrowtail \Sigma_{st}$ and a canonical isomorphism $(A \wr B) \wr G \simeq A \wr (B \wr G)$.*

Proof. For $k = 1, \dots, s$ write $S_k = \{(k-1)t+1, \dots, kt\}$. Then $\{1, \dots, st\} = S_1 \sqcup \dots \sqcup S_s$ and we get embeddings $B^s \rightarrowtail \Sigma_{st}$ (where the k^{th} factor permutes S_k) and $A \rightarrowtail \Sigma_{st}$ (where A permutes $\{S_1 \dots, S_s\}$ in the obvious way). Then, since any element of $A \cap B^s$ must map $S_k \xrightarrow{\sim} S_k$ for each k, it is clear that $A \cap B^s = 1$. Further, if $\sigma \in A$ and $\tau \in B^s$ then, taking $i \in S_k$, we have $\sigma\tau\sigma^{-1}(i) \in S_{\sigma\sigma^{-1}(k)} = S_k$, so that $\sigma\tau\sigma^{-1} \in B^s$. Hence $A \leqslant N_{\Sigma_{st}}(B^s)$ and an application of Proposition 2.4 gives us $AB^s = A \ltimes B^s = A \wr B$ as a subgroup of Σ_{st}.

The proof that $(A \wr B) \wr G \simeq A \wr (B \wr G)$ follows on careful checking that the map

$$((a;b_1,\dots,b_s);g_1,\dots,g_{st}) \mapsto (a;(b_1;g_1,\dots,g_t),\dots,(b_s;g_{(s-1)t+1},\dots,g_{st}))$$

is an isomorphism. □

Another useful feature is that the wreath product distributes over the cross product.

Lemma 2.6. *Let $C \leqslant \Sigma_s$ and $D \leqslant \Sigma_t$. Then, viewing $C \times D$ as a subgroup of Σ_{s+t} the map $(C \wr G) \times (D \wr G) \to (C \times D) \wr G$ given by*

$$((\sigma;g_1,\dots,g_s),(\tau;h_1,\dots,h_t)) \mapsto ((\sigma,\tau);g_1,\dots,g_s,h_1,\dots,h_t)$$

is an isomorphism.

Proof. The map is visibly a bijection. Checking that it is a group homomorphism is straightforward, if a little fiddly. □

2.1.7 Classifying spaces

A *topological group* is a group equipped with a Hausdorff topology for which the multiplication and inverse maps are continuous. Given a topological group G (with a CW structure) there is a CW-complex known as the *classifying space of G*, denoted BG, which is formed as the geometric realisation of the nerve of the category $\mathbb{G}$ in which there is just one object with morphisms indexed by elements of G.

The assignment $G \mapsto BG$ is functorial and, for a large class of groups (in particular, all countable groups), we have a homeomorphism $B(G \times H) \simeq BG \times BH$.[1] If G carries the discrete topology, such as when G is finite, then BG is a $K(G,1)$ Eilenberg-MacLane space, that is $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for all $n \neq 1$. Of fundamental importance is the space BS^1 which turns out to be $\mathbb{C}P^\infty$.

We have the following useful result.

Proposition 2.7. *Let G be a topological group. Then the map $conj_g : G \to G, h \mapsto ghg^{-1}$ induces a map $BG \to BG$ which is homotopic to the identity.*

Proof. This is covered in [Seg68, Section 3]. It is a corollary of the fact that for any topological categories $\mathcal{C}$ and $\mathcal{C}'$ and continuous functors $F_1, F_2 : \mathcal{C} \to \mathcal{C}'$, if there is a natural transformation $F : F_0 \to F_1$ then $BF_0, BF_1 : B\mathcal{C} \to B\mathcal{C}'$ are homotopic. Putting $\mathcal{C}, \mathcal{C}' = G$, $F_0 = \mathrm{conj}_g$ and $F_1 = \mathrm{id}_G$ then for any $h \in G$ we have $F_1(h) = ghg^{-1}$ and $F_0(h) = h$ and hence a commutative diagram

$$\begin{tikzcd}
F_1(*) \arrow[r, "g^{-1}"] \arrow[d, "F_1(h)"'] & F_0(*) \arrow[d, "F_0(h)"] \\
F_1(*) \arrow[r, "g^{-1}"] & F_0(*).
\end{tikzcd}$$

Thus we have a natural transformation given by $F_0(*) \xrightarrow{g^{-1}} F_1(*)$ and the result follows. $\square$

2.1.8 The elementary symmetric functions

Let R be a ring. Then Σ_d acts on the power series ring $R[\![x_1, \ldots, x_d]\!]$ by $\tau.x_i = x_{\tau(i)}$ and the ring of invariants is given by $R[\![x_i, \ldots, x_d]\!]^{\Sigma_d} = R[\![\sigma_1 \ldots, \sigma_d]\!]$, where σ_k is known as the k^{th} *elementary symmetric function* and is defined by

$$\sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq d} x_{i_1} \ldots x_{i_k}$$

(so that $\sigma_1 = x_1 + \ldots + x_d,\ \sigma_2 = x_1x_2 + x_1x_3 + \ldots + x_{d-1}x_d, \ldots,\ \sigma_d = x_1 \ldots x_d$).

Letting $N \in \mathbb{N}$, write $q : R[\![x_1, \ldots, x_d]\!] \to R[\![x_1, \ldots, x_d]\!]/(x_1^N, \ldots, x_d^N)$ for the quotient map and identify σ_i with $q(\sigma_i)$ for each i. We have the following lemma.

Lemma 2.8. *Let $N \in \mathbb{N}$. Then the elements $\sigma_1^{\beta_1} \ldots \sigma_d^{\beta_d} \in R[\![x_1, \ldots, x_d]\!]/(x_1^N, \ldots, x_d^N)$ for $\beta_1, \ldots, \beta_d \in \mathbb{N}$ with $0 \leq \beta_1 + \ldots + \beta_d < N$ are linearly independent.*

Proof. Let B be the set $\{(\beta_1, \ldots, \beta_d) \in \mathbb{N}^d \mid 0 \leq \beta_1 + \ldots + \beta_d < N\}$ and suppose that $\sum_{\beta \in B} r_\beta \sigma_1^{\beta_1} \ldots \sigma_d^{\beta_d} = 0$ for some $r_\beta \in R$. Then, for each β and each $1 \leq i \leq d$, the highest power of x_i occurring in the expression $\sigma_1^{\beta_1} \ldots \sigma_d^{\beta_d}$ is no more than $\beta_1 + \ldots + \beta_d < N$. Hence the relation lifts to $R[\![x_1, \ldots, x_d]\!]$ whereby $r_\beta = 0$ for all β. $\square$

Proposition 2.9. *Let $N \in \mathbb{N}$. Then the free R-module $(R[\![x_1, \ldots, x_d]\!]/(x_1^N, \ldots, x_d^N))^{\Sigma_d}$ has basis $B = \{\sigma_1^{\beta_1} \ldots \sigma_d^{\beta_d} \mid 0 \leq \beta_1 + \ldots + \beta_d < N\}$.*

[1] The problem that can arise here is that the topology on $B(G \times H)$ does not, in general, coincide with the product topology on $BG \times BH$. Instead the right hand-side must be given the compactly generated topology (see [Seg68]). They do coincide, however, if BG and BH have countably many cells (see [Hat02, Appendix]).

Proof. Take a non-zero element $y \in (R[\![x_1, \ldots, x_d]\!]/(x_1^N, \ldots, x_d^N))^{\Sigma_d}$. Let A denote the set $\{\alpha \in \mathbb{N}^d \mid 0 \leq \alpha_i < N\}$. Then, for $\alpha \in A$, we write $\mathbf{x}^\alpha$ for $x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ and, using the standard basis for $R[\![x_1, \ldots, x_d]\!]/(x_1^N, \ldots, x_d^N)$, we write $y = \sum_{\alpha \in A} r_\alpha \mathbf{x}^\alpha$ for some $r_\alpha \in R$.

Note that we can define an action of Σ_d on A by $\tau.(\alpha_1, \ldots, \alpha_d) = (\alpha_{\tau^{-1}(1)}, \ldots, \alpha_{\tau^{-1}(d)})$ and, with this action, $\tau.\mathbf{x}^\alpha = \mathbf{x}^{\tau.\alpha}$. Letting $\tau \in \Sigma_d$ then, since $\tau^{-1}.y = y$, we have

$$\sum_{\alpha \in A} r_\alpha \mathbf{x}^\alpha = \tau^{-1}.\sum_{\alpha \in A} r_\alpha \mathbf{x}^\alpha = \sum_{\alpha \in A} r_\alpha \mathbf{x}^{\tau^{-1}.\alpha} = \sum_{\alpha \in A} r_{\tau.\alpha} \mathbf{x}^\alpha.$$

Hence we see that $r_\alpha = r_{\tau.\alpha}$ for all $\alpha \in A$ and all $\tau \in \Sigma_d$. Next, introduce an ordering $\prec$ on the monomials in $(R[[x_1, \ldots, x_d]]/(x_1^N, \ldots, x_d^N))^{\Sigma_d}$ by

$$\begin{aligned} x_1^{\alpha_1} \ldots x_d^{\alpha_d} \succ x_1^{\beta_1} \ldots x_d^{\beta_d} \iff & \alpha_1 > \beta_1 \\ & \text{or } \alpha_1 = \beta_1 \text{ and } \alpha_2 > \beta_2 \\ & \text{or } \alpha_1 = \beta_1, \alpha_2 = \beta_2 \text{ and } \alpha_3 > \beta_3, \text{ etc.} \end{aligned}$$

This is a total ordering on the monomials in $(R[[x_1, \ldots, x_d]]/(x_1^N, \ldots, x_d^N))^{\Sigma_d}$ (the *lexicographical ordering*). Now, let $B = \{\sigma_1^{\beta_1} \ldots \sigma_d^{\beta_d} \mid 0 \leq \beta_1 + \ldots + \beta_d < N\}$. Let $r_m m = r_m \mathbf{x}^\alpha$ be the largest monomial appearing as a summand of y. Note that m is of the form $x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ with $\alpha_1 \geq \ldots \geq \alpha_d$ since otherwise we could find some $\tau \in \Sigma_d$ such that $\tau.m \succ m$ and $\tau.m$ necessarily appears as a summand of y. Now,

$$\begin{aligned} \sigma_1^{\alpha_1 - \alpha_2} \sigma_2^{\alpha_2 - \alpha_3} \ldots \sigma_d^{\alpha_d} &= x_1^{\alpha_1 - \alpha_2} (x_1 x_2)^{\alpha_2 - \alpha_3} \ldots (x_1 \ldots x_d)^{\alpha_d} + \text{ lower terms} \\ &= x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d} + \text{ lower terms} \\ &= m + \text{ lower terms.} \end{aligned}$$

Hence $y - r_m \sigma_1^{\alpha_1 - \alpha_2} \sigma_2^{\alpha_2 - \alpha_3} \ldots \sigma_d^{\alpha_d}$ consists of monomials strictly smaller than m. Since y has a finite number of monomial summands we can continue in this way to get y expressed as a linear sum of elements of B in a finite number of steps. Thus B spans $(R[[x_1, \ldots, x_d]]/(x_1^N, \ldots, x_d^N))^{\Sigma_d}$ and hence, using Lemma 2.8, is a basis. ☐

2.1.9 Nakayama's lemma and related results

In this section we include a few useful results from commutative algebra. We begin with a version of Nakayama's lemma.

Proposition 2.10 (Nakayama's lemma)**.** *Let R be a local ring and M a finitely generated R-module. If I is a proper ideal of R and $M = IM$ then $M = 0$.*

Proof. This is covered in [Mat89]. ☐

We will usually apply a corollary of this result, but first need the following lemma.

Lemma 2.11. *Let R be a ring, I an ideal in R and M an R-module. Then $R/I \otimes_R M \simeq M/IM$.*

Proof. Define a map $f : M \to (R/I) \otimes_R M$ by $f(m) = 1 \otimes m$. Then it is easy to show that $IM \subseteq \ker(f)$ so that f factors through a map $\overline{f} : M/IM \to (R/I) \otimes_R M$. It is then not difficult to check that the map $(R/I) \otimes_R M \to M/IM$, $\overline{a} \otimes m \mapsto \overline{a}.\overline{m}$ is inverse to $\overline{f}$ which gives the result. ☐

Proposition 2.12. *Let R be a local ring, I a proper ideal in R and $\alpha : M \to N$ be a map of finitely generated R-modules. If the induced map $M/IM \to N/IN$ is an isomorphism then α is surjective. Hence, if α is just the inclusion of M in N then $M = N$.*

Proof. The exact sequence $M \xrightarrow{\alpha} N \to N/\alpha(M) \to 0$ induces an exact sequence

$$(R/I) \otimes M \to (R/I) \otimes N \to (R/I) \otimes (N/\alpha(M)) \to 0.$$

Using Lemma 2.11 and the fact that $M/IM \to N/IN$ is an isomorphism we see that $(N/\alpha(M))/I(N/\alpha(M)) = (R/I) \otimes (N/\alpha(M)) = 0$. Hence, $N/\alpha(M) = I(N/\alpha(M))$ and, by Nakayama's lemma, $N/\alpha(M) = 0$ so that $N = \alpha(M)$. $\square$

Let R be a ring and M an R-module. Then an element $x \in R$ is *regular on M* if $x.m = 0$ implies $m = 0$ ($m \in M$). The ordered sequence $x_1, \ldots, x_n$ of elements of R is a *regular sequence on M* if x_1 is regular on M, x_2 is regular on M/x_1M,..., x_n is regular on $M/(x_1, \ldots, x_{n-1})M$ and $M/(x_1, \ldots, x_n)M \neq 0$.

Lemma 2.13. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and $\alpha : M \to N$ a map of finitely generated R-modules. Suppose that $\mathfrak{m} = (x_1, \ldots, x_n)$ and $x_1, \ldots, x_n$ is a regular sequence on both M and N. If the induced map $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism then so is α.*

Proof. By Lemma 2.12 we know that α is surjective, so it remains to show injectivity. Let $K = \ker(\alpha)$. Then, since x_1 is regular on N, M and K, we have a diagram of exact sequences

$$\begin{tikzcd}
x_1K \arrow[r, tail] \arrow[d, tail] & x_1M \arrow[r, two heads, "\alpha"] \arrow[d, tail] & x_1N \arrow[d, tail] \\
K \arrow[r, tail] \arrow[d, two heads] & M \arrow[r, two heads, "\alpha"] \arrow[d, two heads] & N \arrow[d, two heads] \\
K/x_1K \arrow[r] & M/x_1M \arrow[r, two heads] & N/x_1N
\end{tikzcd}$$

and a diagram chase shows that the map $K/x_1K \to M/x_1M$ is injective. Hence we can repeat the process to end up with an exact sequence

$$K/(x_1, \ldots, x_n)K \rightarrowtail M/(x_1, \ldots, x_n)M \twoheadrightarrow N/(x_1, \ldots, x_n)N.$$

But, by our hypothesis, $(x_1, \ldots, x_n) = \mathfrak{m}$ and $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism, so $K/\mathfrak{m}K = 0$. Thus an application of Nakayama's lemma gives $K = 0$ and α is injective. $\square$

Corollary 2.14. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finitely generated R-module. Suppose that $\mathfrak{m} = (x_1, \ldots, x_n)$ and $x_1, \ldots, x_n$ is a regular sequence on M. Then M is free over R.*

Proof. Reduce modulo $\mathfrak{m}$ and choose a basis of the finite dimensional $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$. Lift this basis to get a map $R^d \to M$ for some d which gives a mod-$\mathfrak{m}$ isomorphism. Now applying the previous lemma we find that the map $R^d \to M$ is an isomorphism and M is free over R. $\square$

2.1.10 Regular local rings and related algebra

Given a ring R we define the *Krull dimension* of R to be the supremum over the lengths r of all strictly decreasing chains of prime ideals $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_r$. Of particular interest will be power series rings $R[\![x_1, \ldots, x_k]\!]$ which, if R is Noetherian of Krull dimension n, have Krull dimension $n+k$. Note that if $(R, \mathfrak{m})$ is a local ring then the Krull dimension of R is zero if and only if $\mathfrak{m}$ is the only prime ideal of R.

Lemma 2.15. *Let $(R, \mathfrak{m})$ be a local Noetherian K-algebra for some field K. Suppose that R has Krull dimension 0 and that $R/\mathfrak{m}$ is finite dimensional over K. Then R is finite dimensional over K.*

Proof. Since R is Noetherian it follows that $\mathfrak{m}$ is a finitely generated ideal. Hence, each of the vector spaces $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ are finite dimensional over $R/\mathfrak{m}$ and hence also over K. Further, since R has Krull dimension 0, $\mathfrak{m}$ is the unique prime ideal of R. Thus $\mathrm{nil}(R) = \mathfrak{m}$ so that, in particular, all the generators of $\mathfrak{m}$ are nilpotent. It follows that there is $N \in \mathbb{N}$ such that $\mathfrak{m}^{N+1} = 0$, whereby $\mathfrak{m}^N = \mathfrak{m}^N/\mathfrak{m}^{N+1}$ is also finite dimensional. Thus we find that $R \simeq R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \ldots \oplus \mathfrak{m}^{N-1}/\mathfrak{m}^N \oplus \mathfrak{m}^N$ is finite dimensional over K. ☐

Let $(R, \mathfrak{m})$ be a Noetherian local ring of Krull dimension n. Then the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is called the *embedding dimension* of R. This is equal to the smallest number of elements needed to generate $\mathfrak{m}$ over R and hence $\mathrm{embdim}(R) \geq n$. If $\mathrm{embdim}(R) = n$ then R is called a *regular local ring* and a minimal generating set for $\mathfrak{m}$ is called a *regular system of parameters.* Such a generating set is automatically a regular sequence on R (see [Mat89, Chapter 5]).

Lemma 2.16. *Let $(R, \mathfrak{m})$ be a complete local Noetherian ring. If M is an R-module such that $M/\mathfrak{m}M$ is generated over $R/\mathfrak{m}$ by $\mu_1, \ldots, \mu_r$ and $m_i \in M$ lifts μ_i then M is generated over R by $m_1, \ldots, m_r$. Hence, if $M/\mathfrak{m}M$ is finitely generated over $R/\mathfrak{m}$ then M is finitely generated over R.*

Proof. This is Theorem 8.4 in [Mat89]. ☐

We include the following elementary lemmas for reference later on.

Lemma 2.17. *Let R be a ring, I an ideal in R, A an R-algebra and J an ideal in A. Then the exact sequence*

$$0 \to J \rightarrowtail A \twoheadrightarrow A/J \to 0$$

induces a (right) exact sequence

$$(R/I) \otimes_R J \to (R/I) \otimes_R A \twoheadrightarrow (R/I) \otimes_R (A/J) \to 0.$$

In particular, the map $(R/I) \otimes_R J \to \ker\big((R/I) \otimes_R A \twoheadrightarrow (R/I) \otimes_R (A/J)\big)$ is surjective.

Proof. This is an immediate consequence of the right-exactness of the tensor product. ☐

Corollary 2.18. *Let R be a ring, I an ideal in R, A an R-algebra and $a \in A$. Then reduction modulo I induces a surjective map $Aa/I(Aa) \twoheadrightarrow (A/IA)\overline{a}$, where $\overline{a}$ is the image of a in A/IA.*

Proof. This is a special case of Lemma 2.17. Let $K = \ker(A/IA \twoheadrightarrow (A/Aa)/I(A/Aa))$. Then it is clear that $(A/IA)\overline{a} \subseteq K$ and we must show that $K = (A/IA)\overline{a}$. Take $x \in K \leqslant A/IA$. Then x lifts to $\tilde{x} \in A$ and, writing q for the quotient map $A \to A/Aa$, we have $q(\tilde{x}) \in I(A/Aa)$, say $q(\tilde{x}) = s.y$ for some $s \in I, y \in A/Aa$. Then y lifts to $\tilde{y} \in A$ and $\tilde{x} - s\tilde{y} \in \ker q = Aa$. Reducing mod I we get $x \in (A/IA)\overline{a}$, as required. □

Remark 2.19. It is important to note that, in general, $Aa/I(Aa) \twoheadrightarrow (A/IA)\overline{a}$ is not an isomorphism. As an example, let $R = A = \mathbb{Z}$, $a = p$ and $I = p\mathbb{Z}$. Then $Aa = I$ so that $Aa/I(Aa) = I/I^2 \simeq \mathbb{Z}/p$ whereas $(A/IA)\overline{a} = (I/I^2)p = (\mathbb{Z}/p)p = 0$. In particular note that the composite $Aa/I(Aa) \twoheadrightarrow (A/IA)\overline{a} \rightarrowtail A/IA$ is not injective.

We use the following results from commutative algebra.

Lemma 2.20. *(Chinese Remainder Theorem) Let R be a commutative ring and I and J ideals in R with $1 \in I + J$. Then the map $R/IJ \to R/I \times R/J$, $a \mapsto (a + I, a + J)$ is an isomorphism.*

Proof. Standard algebra (see [Lan02]). □

Lemma 2.21. *Given a ring R and an ideal I in R we have $\mathbb{Q} \otimes (R/I) \simeq (\mathbb{Q} \otimes R)/(\mathbb{Q} \otimes I)$.*

Proof. The quotient map $R \twoheadrightarrow R/I$ induces a surjection $f : \mathbb{Q} \otimes R \to \mathbb{Q} \otimes (R/I)$. Given $r \in \ker(f)$ there is an $N \in \mathbb{N}$ such that $Nr \in R$ and then $f(Nr) = Nf(r) = 0$ so that $Nr \in I$. Thus $r = \frac{1}{N} \otimes Nr \in \mathbb{Q} \otimes I$ and $\ker(f) \subseteq \mathbb{Q} \otimes I$. The reverse inclusion is clear. Thus we have an isomorphism $(\mathbb{Q} \otimes R)/(\mathbb{Q} \otimes I) \xrightarrow{\sim} \mathbb{Q} \otimes (R/I)$, as required. □

We will use the following corollary.

Corollary 2.22. *Let R be a ring and I and J ideals in R with $k \in I + J$ for some $0 \neq k \in \mathbb{Z}$. Then the induced map $\mathbb{Q} \otimes (R/IJ) \to \mathbb{Q} \otimes (R/I) \times \mathbb{Q} \otimes (R/J)$ is an isomorphism.*

Proof. Note that $\mathbb{Q} \otimes (IJ) = (\mathbb{Q} \otimes I)(\mathbb{Q} \otimes J)$ and $\mathbb{Q} \otimes (I + J) = \mathbb{Q} \otimes I + \mathbb{Q} \otimes J$. Now, $1 = \frac{1}{k} \otimes k \in \mathbb{Q} \otimes (I + J) = \mathbb{Q} \otimes I + \mathbb{Q} \otimes J$ so that the Chinese Remainder Theorem applies and we get

$$\begin{tikzcd}
(\mathbb{Q} \otimes R)/(\mathbb{Q} \otimes (IJ)) \arrow[r, "\sim"] \arrow[d, "\wr"'] & (\mathbb{Q} \otimes R)/(\mathbb{Q} \otimes I) \times (\mathbb{Q} \otimes R)/(\mathbb{Q} \otimes J) \arrow[d, "\wr"] \\
\mathbb{Q} \otimes (R/IJ) \arrow[r] & \mathbb{Q} \otimes (R/I) \times \mathbb{Q} \otimes (R/J)
\end{tikzcd}$$

showing that the bottom map is an isomorphism, as claimed. □

2.1.11 Duality algebras

Let A be an algebra over a ring R. Then the group $\mathrm{Hom}_R(A, R)$ is an A-module via the action $(a.\phi)(b) = \phi(ab)$ and we say that A is a *duality algebra* if there is an A-module isomorphism $\Theta : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$. Note that such an isomorphism will be determined by $\theta = \Theta(1)$. Thus A is a duality algebra if and only if there is a R-linear map $\theta : A \to R$ such that the map $A \to \mathrm{Hom}_R(A, R)$, $a \mapsto a.\theta$ is an isomorphism of R-modules. Such a θ is known as a *Frobenius form.*

Lemma 2.23. *Let A be a duality algebra over R and θ a Frobenius form on A. Let I be any ideal in A. Then if $a \in A$ and $\theta(aI) = 0$ we have $aI = 0$. Hence*

$$ann_A(I) = \{a \in A \mid \theta(aI) = 0\}.$$

Proof. Let $s \in I$. Then $\theta(as) = 0$. Now, for any $b \in A$ we have $\theta(bas) = \theta(a(bs)) \in \theta(aI) = 0$. Thus $(as).\theta$ is the zero map and it follows that $as = 0$. Hence a annihilates I, as claimed. □

Corollary 2.24. *Let A be a duality algebra over R and suppose that P is an R-module summand in A. Then $ann_A(P)$ is a summand in A.*

Proof. Write $A = P \oplus A/P$ and so $\mathrm{Hom}_R(A, R) = \mathrm{Hom}_R(P, R) \oplus \mathrm{Hom}_R(A/P, R)$. Put $Q = \Theta^{-1}(\mathrm{Hom}_R(A/P, R))$. Then Q is a summand in A and, with the notation of the opening paragraph,

$$\begin{aligned} x \in Q &\iff \Theta(x) \in \mathrm{Hom}_R(A/P, R) \\ &\iff (x.\theta)(P) = 0 \\ &\iff \theta(xP) = 0 \\ &\iff x \in \mathrm{ann}_A(P). \end{aligned}$$

Thus $Q = \mathrm{ann}_A(P)$ is a summand in A. □

Now suppose that K is a field and A is a local and finite-dimensional duality algebra over K. Recall from Section 2.1.1 that the socle of A is defined to be the annihilator of its maximal ideal. We will shortly have a useful characterisation of such duality algebras in terms of the dimension of their socles, but first we need a couple of lemmas.

Lemma 2.25. *Let K be field, A a finite-dimensional K-algebra and I an ideal in A. Then A is a local ring with maximal ideal I if and only if A/I is a field and $I^N = 0$ for some N.*

Proof. First suppose that A/I is a field and $I^N = 0$ for some N. Then I is a maximal ideal and $I \subseteq \mathrm{nil}(A)$. Further, since $\mathrm{nil}(A)$ is equal to the intersection of all the prime ideals of A, we have $I \subseteq \mathrm{nil}(A) \subseteq \mathrm{rad}(A) \subseteq I$. It follows that I is the unique maximal ideal of A and A is local.

For the converse, since A is a finite-dimensional vector space it is Artinian and hence all prime ideals are maximal (see, for example, [Mat89, pg 30]). Thus if A is local with maximal ideal I we have $I = \mathrm{rad}(A) = \mathrm{nil}(A)$ so that every element of I is nilpotent. Since the descending chain of vector spaces $I \geqslant I^2 \geqslant I^3 \geqslant \dots$ must be eventually constant it follows that we must have $I^N = 0$ for some N. □

Lemma 2.26. *Let K be field and $(A, \mathfrak{m})$ be a finite-dimensional local K-algebra and suppose that $soc(A)$ is one-dimensional over $A/\mathfrak{m}$. Then if I is any non-trivial ideal of A we have $soc(A) \subseteq I$.*

Proof. By Lemma 2.25 we must have $\mathfrak{m}^N = 0$ for some N. Thus we have a descending chain

$$I \geqslant \mathfrak{m}I \geqslant \mathfrak{m}^2 I \geqslant \dots \geqslant \mathfrak{m}^N I = 0$$

and hence there exists $t \geq 0$ such that $\mathfrak{m}^{t+1}I = 0$ but $\mathfrak{m}^t I \neq 0$. Since $\mathfrak{m}.(\mathfrak{m}^t I) = \mathfrak{m}^{t+1} I = 0$ we see that $\mathfrak{m}^t I$ is a non-zero $A/\mathfrak{m}$-subspace of the one-dimensional vector space $\mathrm{soc}(A)$. Thus we have $\mathrm{soc}(A) = \mathfrak{m}^t I \leqslant I$, as required. □

We are now able to prove some useful results.

Proposition 2.27. *Let K be field and $(A, \mathfrak{m})$ a finite-dimensional local K-algebra such that the composition $K \to A \to A/\mathfrak{m}$ is an isomorphism. Then A is a duality algebra if and only if $soc(A)$ has dimension one over K.*

Proof. Let θ be a Frobenius form on A and let $\Theta : A \xrightarrow{\sim} \mathrm{Hom}_K(A, K)$ be the associated isomorphism of A-modules. The inclusion $\mathfrak{m} \hookrightarrow A$ gives us a splitting $A \simeq (A/\mathfrak{m}) \oplus \mathfrak{m} \simeq K \oplus \mathfrak{m}$ of K-vector spaces. If $a \in \mathrm{soc}(A)$ with $\theta(a) = 0$ then, since a annihilates $\mathfrak{m}$, we have $Aa = (K \oplus \mathfrak{m})a = Ka$ and so $\theta(Aa) = \theta(Ka) = K\theta(a) = 0$. It follows that $\Theta(a) : A \to K$ is the zero map and hence, since Θ is an isomorphism, $a = 0$. Thus $\theta : \mathrm{soc}(A) \to K$ is an injective map of vector spaces meaning $\dim_K(\mathrm{soc(A)}) \leq 1$. Letting t be the largest non-zero power of $\mathfrak{m}$ (necessarily finite) we see that $0 < \mathfrak{m}^t \leqslant \mathrm{soc}(A)$ so that $\mathrm{soc}(A)$ is non-zero and therefore that $\dim_K(\mathrm{soc}(A)) = 1$, as required.

Conversely, suppose that $\dim_K(\mathrm{soc}(A)) = 1$. Let $0 \neq v \in \mathrm{soc}(A)$ so that $\mathrm{soc}(A) = Kv$. Extend $\{v\}$ to a basis $\{v, v_1, \ldots, v_{d-1}\}$ for A over K and let $\phi : A \to K$ be the linear map sending $v \mapsto 1$ and $v_i \mapsto 0$ for $i = 1, \ldots, d-1$. We show that ϕ is a Frobenius form.

Let $\Phi : A \to \mathrm{Hom}_K(A, K)$ be the K-linear map defined by $\Phi(a) = a.\phi$. Hence, if $a \in A$ and $\Phi(a) = 0$ then $\phi(aA) = 0$. But aA is an ideal in A and every non-trivial ideal contains $\mathrm{soc}(A)$ by Lemma 2.26. Thus, since $\phi(\mathrm{soc}(A)) = K \neq 0$, we must have $aA = 0$ and hence $a = 0$, so Φ is injective. Since $\dim_K(\mathrm{Hom}_K(A, K)) = \dim_K(A)$ (and both are finite) it follows that Φ must be an isomorphism of vector spaces. Thus ϕ is a Frobenius form on A. □

Proposition 2.28. *Let A and K be as above. Suppose char$(K) = p$ and let G be a group of K-algebra automorphisms of A such that $p \nmid |G|$. Then if A admits a G-invariant Frobenius form its restriction is a Frobenius form on A^G and thus A^G has duality over K.*

Proof. As in the proof of Lemma 2.27 we have $A \simeq K \oplus \mathfrak{m}$ as K-modules. Since G acts by K-algebra automorphisms we have $A^G = (K \oplus \mathfrak{m})^G = K \oplus \mathfrak{m}^G$ so that $A^G/\mathfrak{m}^G \simeq K$. Further, since $\mathfrak{m}^G \subseteq \mathfrak{m}$ and $\mathfrak{m}^N = 0$ for some N we have $(\mathfrak{m}^G)^N = 0$. Hence, by Lemma 2.25, A^G is local and $\mathfrak{m}^G$ its maximal ideal.

Let $\theta : A \to K$ be the G-invariant Frobenius form on A, so that $\theta(g.a) = \theta(a)$ for all $a \in A$ and $g \in G$, and let $\Theta : A \xrightarrow{\sim} \mathrm{Hom}_K(A, K)$ be the associated isomorphism of vector spaces. Let Φ be the composite $A^G \rightarrowtail A \xrightarrow{\Theta} \mathrm{Hom}_K(A, K) \xrightarrow{\mathrm{res}} \mathrm{Hom}_K(A^G, K)$. Then Φ is an A^G-linear map of finite dimensional K-vector spaces of the same dimension.

Now, define an A^G-linear map $r : A \to A^G$ by $r(b) = \frac{1}{|G|}\sum_{g\in G} g.b$. Then, for $a \in A^G$ we have $ar(b) = a\frac{1}{|G|}\sum_{g\in G} g.b = \frac{1}{|G|}\sum_{g\in G} g.(ab) = r(ab)$ so that

$$(\Phi(a) \circ r)(b) = ((a.\theta) \circ r)(b) = \theta(ar(b)) = \theta(r(ab)).$$

But $\theta \circ r = \theta$ since θ is G-invariant. Hence we have $\Phi(a) \circ r = \Theta(a)$ as maps from $A \to K$. Thus, if $\Phi(a) = 0$ we have $\Theta(a) = \Phi(a) \circ r = 0$ so that $a = 0$. Hence Φ is injective and therefore an isomorphism. Since Φ is A^G-linear it follows that A^G has duality over K with Frobenius form $\Phi(1) = \theta|_{A^G}$. □

2.1.12 The p-divisibility of $k^s - 1$.

For this section we will assume that p is an odd prime and let k be any integer. For reasons that should become clear later we aim to get a good understanding of the p-divisibility of $k^s - 1$ for varying $s \in \mathbb{N}$. That is, in the notation of Section 2.1.2, we are looking to calculate $v_p(k^s - 1)$. Note that if k is divisible by p then $v_p(k^s - 1) = 0$ for all s. Hence we can assume that k is coprime to p and start with the case where $k = 1 \bmod p$.

Lemma 2.29. *Suppose $v_p(k-1) = v > 0$ and take $s \geq 1$ with $(s, p) = 1$. Then $v_p(k^s-1) = v$.*

Proof. Write $k = 1 + ap^v$ with $(a, p) = 1$. If $s = 1$ the result is clear. Otherwise, $s > 1$ and for all $1 < i \leq s$ we have $(p^v)^i = p^{iv} = p^{v+1}.p^{(i-1)v-1}$ which is divisible by p^{v+1} since $(i-1)v - 1 \geq 0$. Then

$$\begin{aligned} k^s &= (1 + ap^v)^s \\ &= 1 + s.ap^v + \sum_{i=2}^{s} \binom{s}{i} (ap^v)^i \\ &= 1 + s.ap^v + p^{v+1}.b \end{aligned}$$

for some b. Hence $k^s - 1 = p^v(sa + pb)$ whereby $v_p(k^s - 1) = v$ since $p \nmid sa$. $\square$

Lemma 2.30. *For $0 < i < p$ we have $v_p\left(\binom{p}{i}\right) = 1$.*

Proof. We have $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ so that $i!(p-i)!\binom{p}{i} = p!$. Since $v_p(p!) = 1$ and $v_p(i!) = v_p((p-i)!) = 0$ we see that $v_p\binom{p}{i} = 1$, as required. $\square$

Corollary 2.31. *Let $v_p(k-1) = v > 0$. Then $v_p(k^p - 1) = v + 1$.*

Proof. Writing $k = 1 + ap^v$ with $(a, p) = 1$ we have

$$\begin{aligned} k^p = (1 + ap^v)^p &= 1 + p.ap^v + \left(\sum_{i=2}^{p-1} \binom{p}{i} (ap^v)^i\right) + (ap^v)^p \\ &= 1 + ap^{v+1} + \left(\sum_{i=2}^{p-1} \binom{p}{i} a^i p^{iv}\right) + a^p p^{vp}. \end{aligned}$$

But for $2 \leq i \leq p - 1$ we have

$$\begin{aligned} v_p\left(\binom{p}{i} a^i p^{iv}\right) &= v_p\binom{p}{i} + v_p(ap^{iv}) \\ &= 1 + iv \\ &\geq v + 2 \end{aligned}$$

since $v \geq 1$ and $i \geq 2$. Similarly, $v_p(a^p p^{vp}) = vp \geq v + 2$ since $v \geq 1$ and $p > 2$. Hence we have

$$k^p - 1 = ap^{v+1} + p^{v+2}b = p^{v+1}(a + pb)$$

for some b and the result follows. $\square$

Assembling the above results we get the following.

Lemma 2.32. *Let $v_p(k-1) = v > 0$. Then $v_p(k^s - 1) = v + v_p(s)$.*

Proof. Write $s = ap^w$ with $(a, p) = 1$. Noting that $v_p(k^s - 1) > 0$ for any s (since $k - 1$ divides $k^s - 1$) we have

$$\begin{aligned} v_p(k^s - 1) &= v_p(k^{ap^w} - 1) \\ &= v_p((k^{p^w})^a - 1) \\ &= v_p(k^{p^w} - 1) \quad \text{by Lemma 2.29} \\ &= v_p(k - 1) + w \end{aligned}$$

by repeated use of Corollary 2.31. Thus $v_p(k^s - 1) = v + v_p(s)$, as claimed. $\square$

We can now deal with the general case.

Proposition 2.33. *Let a be the order of k in $(\mathbb{Z}/p)^\times$. Then*

$$v_p(k^s - 1) = \begin{cases} 0 & \textit{if } a \nmid s \\ v_p(k^a - 1) + v_p(s) & \textit{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} v_p(k^s - 1) > 0 \quad &\Longleftrightarrow \quad p \mid k^s - 1 \\ &\Longleftrightarrow \quad k^s = 1 \bmod p \\ &\Longleftrightarrow \quad a \mid s. \end{aligned}$$

Thus $v_p(k^s - 1) = 0$ when $a \nmid s$. If $a \mid s$, write $k' = k^a$. Then $v_p(k' - 1) > 0$ and Lemma 2.32 gives us

$$v_p(k^s - 1) = v_p((k')^{(s/a)} - 1) = v_p(k' - 1) + v_p(s/a) = v_p(k^a - 1) + v_p(s)$$

where we have used the fact that $v_p(s/a) = v_p(s)$ since $a|p - 1$ and so is coprime to p. $\square$

Chapter 3

The p-local structure of the symmetric and general linear groups

There is a close connection between the Sylow p-subgroups of Σ_d and those of $GL_d(K)$, where K is a finite field of characteristic different to p. We begin with an analysis of the former.

3.1 The Sylow p-subgroups of the symmetric groups

The work here is well-known; similar expositions can be found in [AM04] and [Hal76].

Lemma 3.1. *Take $d \in \mathbb{N}$ and write $d = \sum_{i=0}^{r} a_i p^i$ with $0 \le a_i < p$. Then*

$$v_p(d!) = \frac{d - \sum_i a_i}{p-1}$$

where v_p is the p-adic valuation of Section 2.1.2.

Proof. Noting that the integer part of $\frac{d}{p^j}$ is just $\sum_{i=j}^{r} a_i p^{i-j}$ and that, by the usual arguments[1], $v_p(d!) = \sum_{j=1}^{\infty} \left\lfloor \frac{d}{p^j} \right\rfloor$ we get

$$v_p(d!) = \sum_{j=1}^{\infty} \left\lfloor \frac{d}{p^j} \right\rfloor = \sum_{j=1}^{\infty} \sum_{i=j}^{r} a_i p^{i-j} = \sum_{k=1}^{r} \left(a_k \sum_{l=0}^{k-1} p^l \right) = \sum_{k=1}^{r} a_k \left(\frac{p^k - 1}{p-1} \right).$$

But

$$\sum_{k=1}^{r} a_k \left(\frac{p^k - 1}{p-1} \right) = \frac{\sum_{k=1}^{r} a_k p^k - \sum_{k=1}^{r} a_k}{p-1} = \frac{(d - a_0) - \sum_{k=1}^{r} a_k}{p-1} = \frac{d - \sum_i a_i}{p-1},$$

and we have the claimed result. □

Corollary 3.2. *For any $k > 0$ we have $v_p(p^k!) = (p^k - 1)/(p-1)$.*

[1] There are $\left\lfloor \frac{d}{p} \right\rfloor$ terms in the sequence $1, 2, 3 \ldots, d$ divisible by p, $\left\lfloor \frac{d}{p^2} \right\rfloor$ terms divisible by p^2 and so on.

Proof. Writing $p^k = 1.p^k$ the result is immediate from the preceding lemma. $\square$

Proposition 3.3. *Let $C_p = \langle \gamma_p \rangle \leqslant \Sigma_p$ be the cyclic group of order p generated by the standard p-cycle. Then, for any $k \geq 1$, the k-fold wreath product $C_p \wr \ldots \wr C_p$ is a Sylow p-subgroup of Σ_{p^k}.*

Proof. We prove this by induction on k. The result is clear for $k = 1$ since $v_p(|\Sigma_p|) = v_p(p!) = 1 = v_p(C_p)$. Next suppose that the k-fold wreath product $P_k = C_p \wr \ldots \wr C_p$ is a Sylow p-subgroup of Σ_{p^k}. Then an application of Lemma 2.5 shows that $P_{k+1} = C_p \wr P_k$ is a subgroup of $\Sigma_{p.p^k} = \Sigma_{p^{k+1}}$. Noting that $|P_{k+1}| = p|P_k|^p$, using Corollary 3.2 we have

$$\begin{aligned} v_p(|P_{k+1}|) = 1 + p.v_p(|P_k|) = 1 + p.v_p(p^k!) &= 1 + p.\frac{p^k - 1}{p - 1} \\ &= \frac{p - 1 + p^{k+1} - p}{p - 1} \\ &= v_p(p^{k+1}!) \end{aligned}$$

so that P_{k+1} is a Sylow p-subgroup of $\Sigma_{p^{k+1}}$. $\square$

Proposition 3.4. *Let $d \in \mathbb{N}$ and write $d = \sum_{i=0}^{r} a_i p^i$. By partitioning $\{1, \ldots, d\}$ appropriately, there is an embedding $\prod_i (\Sigma_{p^i})^{a_i} \rightarrowtail \Sigma_d$ and any Sylow p-subgroup of $\prod_i (\Sigma_{p^i})^{a_i}$ is a Sylow p-subgroup of Σ_d. In particular, $Syl_p(\Sigma_d)$ is a product of iterated wreath products of C_p.*

Proof. We partition d as

$$d = \underbrace{1 + \ldots + 1}_{a_0 \text{ times}} + \underbrace{p + \ldots + p}_{a_1 \text{ times}} + \ldots + \underbrace{p^r + \ldots + p^r}_{a_r \text{ times}}.$$

This induces the required embedding of $\prod_i (\Sigma_{p^i})^{a_i}$ in Σ_d. Now, using Lemma 3.1 and Corollary 3.2, we get

$$\begin{aligned} v_p\left(|\prod_i (\Sigma_{p^i})^{a_i}|\right) = v_p\left(\prod_i |\Sigma_{p^i}|^{a_i}\right) &= \sum_i a_i v_p(p^i!) \\ &= \sum_i a_i (p^i - 1)/(p - 1) \\ &= \frac{\sum_i a_i p^i - \sum_i a_i}{p - 1} \\ &= \frac{d - \sum_i a_i}{p - 1} \\ &= v_p(d!) \\ &= v_p(|\Sigma_d|). \end{aligned}$$

Thus $\mathrm{Syl}_p(\Sigma_d) \simeq \mathrm{Syl}_p(\prod_i (\Sigma_{p^i})^{a_i}) \simeq \prod_i \mathrm{Syl}_p(\Sigma_{p^i})^{a_i}$ which is of the form claimed using Proposition 3.3. $\square$

3.1.1 The normalizer of $\mathrm{Syl}_p(\Sigma_p)$

Here we consider the Sylow p-subgroup $C_p = \langle \gamma_p \rangle$ of Σ_p.

Lemma 3.5. *There is an embedding $\phi : Aut(C_p) \rightarrowtail \Sigma_p$ such that for all $\alpha \in Aut(C_p)$ we have $\alpha.\gamma_p = \phi(\alpha)\gamma_p\phi(\alpha)^{-1}$.*

Proof. We begin by noting that $\mathrm{Aut}(C_p) \simeq (\mathbb{Z}/p)^\times$ where $s.\gamma_p = \gamma_p^s$. Let $1 \neq s \in (\mathbb{Z}/p)^\times$. Then $s+1, s^2+1, \ldots, s^{p-1}+1$ are all distinct modulo p and (writing p for $0 \in \mathbb{Z}/p$) we have a $(p-1)$-cycle $(s+1 \;\; s^2+1 \;\; \ldots \;\; s^{p-1}+1)$. We define a map $\phi : \mathrm{Aut}(C_p) \to \Sigma_p$ by $\phi(s) = (s+1 \;\; s^2+1 \;\; \ldots \;\; s^{p-1}+1)$. Then it is straightforward to check that this gives an embedding of $\mathrm{Aut}(C_p)$ in Σ_p. Further, the action of $\mathrm{Aut}(C_p)$ on C_p is now given by conjugation, that is $\phi(s)\gamma_p\phi(s)^{-1} = \gamma_p^s = s.\gamma_p$. ∎

Corollary 3.6. *With the embedding of Lemma 3.5 we have $N_{\Sigma_p}(C_p) = Aut(C_p) \ltimes C_p$.*

Proof. By Lemma 3.5 we can view $\mathrm{Aut}(C_p)$ as a subgroup of Σ_p and, further, $\mathrm{Aut}(C_p) \leqslant N_{\Sigma_p}(C_p)$. Since $\mathrm{Aut}(C_p) \cap C_p = 1$ we have $\mathrm{Aut}(C_p).C_p = \mathrm{Aut}(C_p) \ltimes C_p$ by Proposition 2.4. Thus, as C_p is normal in $\mathrm{Aut}(C_p) \ltimes C_p$, it remains to show that $N_{\Sigma_p}(C_p) \subseteq \mathrm{Aut}(C_p) \ltimes C_p$.

Take $\sigma \in N_{\Sigma_p}(C_p)$. Then $\sigma\gamma_p\sigma^{-1} = \gamma_p^s$ for some s and therefore there is $\tau \in \mathrm{Aut}(C_p)$ with $\sigma\gamma_p\sigma^{-1} = \tau\gamma_p\tau^{-1}$ so that $(\tau^{-1}\sigma)\gamma_p(\tau^{-1}\sigma)^{-1} = \gamma_p$. But, remembering that $\gamma_p = (1 \ldots p)$, this identity means that $(1 \ldots p) = ((\tau^{-1}\sigma)(1) \ldots (\tau^{-1}\sigma)(p))$ whereby we must have $(\tau^{-1}\sigma)(a) = a + k$ for some k; that is $\tau^{-1}\sigma = \gamma_p^k$ so that $\sigma = \tau.\gamma_p^k \in \mathrm{Aut}(C_p).C_p = \mathrm{Aut}(C_p) \ltimes C_p$. ∎

3.2 The Sylow p-subgroups of the finite general linear groups

Let K be a finite field. Let $T_d = (K^\times)^d$ be the maximal torus of $GL_d(K)$ and recall from Section 2.1.5 the embedding of Σ_d as a subgroup of $GL_d(K)$ given by $\sigma \mapsto (\sigma_{ij})$, where

$$\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in the structure of $N_d = \Sigma_d T_d$, that is the set

$$\{g \in GL_d(K) \mid g = \sigma(b_1, \ldots, b_d) \text{ for some } \sigma \in \Sigma_d \text{ and some } (b_1, \ldots, b_d) \in T_d\}.$$

We will show that this is a subgroup of $GL_d(K)$ and that, whenever $v_p(|K^\times|) > 0$, it contains one of $GL_d(K)$'s Sylow p-subgroups.

Lemma 3.7. *Let $(b_1, \ldots, b_d) \in T_d$ and $\sigma \in \Sigma_d$. Then*

$$\sigma(b_1, \ldots, b_d)\sigma^{-1} = (b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(d)})$$

and hence $\Sigma_d \leqslant N_{GL_d(K)}(T_d)$.

Proof. It is a straight forward calculation to check that

$$(\sigma(b_1, \ldots, b_d))_{ij} = \begin{cases} b_j & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases} = ((b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(d)})\sigma)_{ij}.$$ ∎

Corollary 3.8. *N_d is a subgroup of $GL_d(K)$ isomorphic to $\Sigma_d \ltimes T_d$.*

Proof. Since $\Sigma_d \cap T_d = 1$, this is a straight application of Proposition 2.4. ∎

Corollary 3.9. *The map $\Sigma_d \wr K^\times \to N_d$ given by $(\sigma; b_1, \ldots, b_d) \mapsto \sigma(b_1, \ldots, b_d)$ is an isomorphism of groups.*

Proof. Follows immediately from 3.7 and 3.8. $\square$

We have the following alternative characterisation of N_d.

Proposition 3.10. *N_d is the normalizer of T_d in $GL_d(K)$ with associated Weyl group Σ_d.*

Proof. Let $g \in N_{GL_d(K)}(T_d)$ and choose $a \in K^\times$ with $a \neq 1$. Take any $1 \leq s \leq d$ and define $e_s = (a, \ldots, a, 1, a, \ldots, a)$ with 1 in the s^{th} place. Write $ge_sg^{-1} = (b_1, \ldots, b_d)$. Then $ge_s = (b_1, \ldots, b_d)g$. By consideration of the $(i,k)^{\text{th}}$ entry we get the equations $g_{is} = b_i g_{is}$ and $g_{ik}a = b_i g_{ik}$ for $k \neq s$. Now, since g is invertible we can find i with $g_{is} \neq 0$. Then $g_{is} = b_i g_{is}$ whereby $b_i = 1$. Hence, for $k \neq s$, we have $g_{ik} = g_{ik}a$ so that, since $a \neq 1$, we get $g_{ik} = 0$.

Summarising, for any s there is $i = i(s)$ such that $g_{is} \neq 0$ and $g_{ik} = 0$ for all $k \neq s$. Since g is invertible each $i(s)$ must be distinct, that is i is a permutation of $\{1, \ldots, d\}$. It follows that $g \in \Sigma_d T_d = N_d$. Thus $N_{GL_d(K)}(T_d) \leqslant N_d \leqslant N_{GL_d(K)}(T_d)$ so we have equality.

For the Weyl group, it is not hard to show that the centralizer of T_d in $GL_d(K)$ is just T_d by a similar calculation to the above so that

$$W_{GL_d(K)}(T_d) = N_{GL_d(K)}(T_d)/C_{GL_d(K)}(T_d) = N_d/T_d = \Sigma_d. \qquad \square$$

Lemma 3.11. *Let l be a prime different to p and let $q = l^r$ for some r. Let a be the order of q in $(\mathbb{Z}/p)^\times$ and put $m = \left\lfloor \frac{d}{a} \right\rfloor$. Then*

$$v_p(|GL_d(\mathbb{F}_q)|) = mv_p(q^a - 1) + v_p(m!).$$

Proof. By the fact that $GL_d(\mathbb{F}_q)$ consists of all $d \times d$ matrices of maximal rank, there are $q^d - 1$ choices for the first column, $q^d - q$ choices for the second and so on. Hence we get

$$\begin{aligned} |GL_d(\mathbb{F}_q)| &= (q^d - 1)(q^d - q) \ldots (q^d - q^{d-1}) \\ &= q^{1+\ldots+(d-1)}(q^d - 1)(q^{d-1} - 1) \ldots (q - 1). \end{aligned}$$

Thus, since q is coprime to p, using Proposition 2.33 we have

$$\begin{aligned} v_p(|GL_d(\mathbb{F}_q)|) &= v_p((q^d - 1)(q^{d-1} - 1) \ldots (q - 1)) \\ &= \sum_{s=1}^{d} v_p(q^s - 1) \\ &= \sum_{k=1}^{m} (v_p(q^a - 1) + v_p(ka)) \\ &= mv_p(q^a - 1) + v_p(m!), \end{aligned}$$

as claimed, where we have used the fact that $v_p(ka) = v_p(k)$ since a is coprime to p. $\square$

Proposition 3.12. *Let K be a finite field such that $v_p(|K^\times|) > 0$. Let P_0 be a Sylow p-subgroup of Σ_d and $P_1 = \{a \in K^\times \mid a^{p^k} = 1 \text{ for some } k\}$ be the p-part of $K^\times$. Let P be the image of $P_0 \wr P_1$ in N_d. Then P is a Sylow p-subgroup of $GL_d(K)$.*

Proof. Choose an isomorphism $K \simeq \mathbb{F}_q$, where $q = l^r$ for some prime l necessarily different from p. Applying Lemma 3.11 then gives $v_p(|GL_d(K)|) = dv_p(q-1) + v_p(d!)$. On the other hand, since P_0 and P_1 are Sylow p-subgroups of Σ_d and $K^\times$ respectively, we have $v_p(|P_0|) = v_p(d!)$ and $v_p(|P_1|) = v_p(q-1)$ so that

$$v_p(|P|) = v_p(|P_0 \wr P_1|) = v_p(|P_0|) + v_p(|P_1^d|) = v_p(d!) + dv_p(q-1)$$

showing that P is indeed a Sylow p-subgroup of $GL_d(K)$. □

Remark 3.13. Suppose that $v_p(|K^\times|) > 0$. Note that for $d < p$ we have $v_p(d!) = 0$ so that $\mathrm{Syl}_p(\Sigma_d) = 1$ and hence $\mathrm{Syl}_p(GL_d(K)) = \mathrm{Syl}_p(T_d)$, which is abelian. For $d \geq p$ the Sylow p-subgroup P_0 of Σ_d is non trivial and the corresponding Sylow p-subgroup of $GL_d(K)$ is no longer abelian.

The Sylow p-subgroups of $GL_d(K)$ for $v_p(|K^\times|) = 0$ are harder to get a handle on, although we do have the following result, valid when $d < p$.

Proposition 3.14. *Let $d < p$ and let $K = \mathbb{F}_q$, where $q = l^r$ for some prime l different to p and some r. Let a be the order of q in $(\mathbb{Z}/p)^\times$ and put $m = \lfloor \frac{d}{a} \rfloor$. Choose a basis for $\mathbb{F}_{q^a}$ over $\mathbb{F}_q$ to get an embedding $\mathbb{F}_{q^a} \rightarrowtail GL_a(\mathbb{F}_q)$. Then, using this embedding, we can view $(\mathbb{F}_{q^a}^\times)^m$ as a subgroup of $GL_d(\mathbb{F}_q)$ and, writing $P_2 = Syl_p(\mathbb{F}_{q^a}^\times)$, we find that P_2^m is a Sylow p-subgroup of $GL_d(\mathbb{F}_q) = GL_d(K)$.*

Proof. We have $v_p(|GL_d(K)|) = mv_p(q^a - 1) + v_p(m!) = v_p((q^a-1)^m) = v_p(|(\mathbb{F}_{q^a}^\times)^m|)$. □

3.3 The Abelian p-subgroups of $GL_p(K)$ for $v_p(|K^\times|) > 0$.

Here we specialise to the general linear groups of dimension p. To begin with we look at the abelian p-subgroups of $N_p = \Sigma_p.T_p \leqslant GL_p(K)$ which, by earlier work, contains a Sylow p-subgroup of $GL_p(K)$. Let $\pi : N_p \twoheadrightarrow \Sigma_p$ denote the projection $\sigma(b_0, \ldots, b_{p-1}) \mapsto \sigma$. For the remainder of this chapter we will omit the subscripts and write N for N_p and T for T_p. We will also write $v = v_p(|K^\times|)$, which will be positive by assumption.

Lemma 3.15. *Let A be a p-subgroup of N. Then either $A \leqslant T$ or $\pi(A) \leq \Sigma_p$ is cyclic of order p, generated by a p-cycle.*

Proof. By Proposition 3.9 there is an exact sequence of groups $T \hookrightarrow N \overset{\pi}{\twoheadrightarrow} \Sigma_p$ and this shows that either $A \leqslant \ker(\pi) = T$ or $\pi(A)$ is a non-trivial p-subgroup of Σ_p. Since $v_p(|\Sigma_p|) = v_p(p!) = 1$ the latter case means $\pi(A)$ is cyclic of order p. To see that $\pi(A)$ is generated by a p-cycle, take a generator of $\pi(A)$; the cycle decomposition of this generator contains only p-cycles (by consideration of its order). As the cycles are disjoint there can only be one. □

Lemma 3.16. *Let $a \in N$ with $\pi(a) \neq 1$ in Σ_p. Then if $b \in T$ we have $conj_a(b) = conj_{\pi(a)}(b)$. Hence $conj_a$ permutes the coordinates of T by a non-trivial cyclic permutation.*

Proof. For the first statement, using Lemma 3.15, write $\pi(a) = \sigma$ for some non-trivial p-cycle σ so that $a = \sigma a'$ for some $a' \in \ker(\pi) = T$. Then for $b \in T$ we have

$$\begin{aligned} \mathrm{conj}_a(b) &= \sigma a' b (a')^{-1} \sigma^{-1} \\ &= \sigma b \sigma^{-1} \\ &= \mathrm{conj}_{\pi(a)}(b) \end{aligned}$$

where the second equality uses the fact that T is abelian. By Lemma 3.7 we see that conj_a permutes the coordinates of T by a non-trivial cyclic permutation, as required. $\square$

Definition 3.17. Let $\Delta \simeq K^\times$ denote the diagonal subgroup of T, and let $\Delta_p \leqslant \Delta$ denote the p-elements of Δ. Note that Δ_p is cyclic of order p^v.

Lemma 3.18. *Let A be an abelian p-subgroup of N with $A \to \Sigma_p$ non-trivial. Then we have $A \cap T \leqslant \Delta_p$.*

Proof. Let $a \in A$ with $\pi(a) \neq 1$. Then, since A is abelian, we get $\mathrm{conj}_a(a') = a'$ for all $a' \in A$. If $a' \in A \cap T$, then since $\pi(a) \neq 1$ we can use Lemma 3.16 to see that all coordinates of a' must be equal. That is, we must have $a' \in \Delta$. Since A is a p-group we get $A \cap T \leqslant \Delta_p$, as required. $\square$

Corollary 3.19. *Let A be an abelian p-subgroup of N with $A \to \Sigma_p$ non-trivial and let $a \in A$ map to a generator of $\pi(A)$. Then $a^p \in \Delta_p$.*

Proof. We have $\pi(a^p) = \pi(a)^p = 1$ so that $a^p \in \ker(\pi) = T$. Thus $a^p \in T \cap A \leqslant \Delta_p$ by Lemma 3.18. $\square$

Corollary 3.20. *Let A be an abelian p-subgroup of N with $A \to \Sigma_p$ non-trivial and let $a \in A$ map to a generator of $\pi(A)$. Then $A \leq \langle a \rangle.\Delta_p$.*

Proof. By Corollary 3.19 we have $a^p \in \Delta_p$. Now, let $a' \in A$ and write $\pi(a') = \pi(a)^k$ for some $0 \leq k < p$. Then $a^{-k}a' \in \ker(\pi) \cap A \leqslant \Delta_p$ so $a' \in \langle a \rangle.\Delta_p$. Thus we have $A \leqslant \langle a \rangle.\Delta_p$. Note that $\langle a \rangle.\Delta_p$ is a subgroup of N since Δ_p is contained in the centre of N. $\square$

We are now ready to give a coarse classification of the abelian p-subgroups of N.

Proposition 3.21. *Let A be an abelian p-subgroup of N. Then either*

1. *$A \leqslant T$,*
2. *$\pi(A)$ is non-trivial and A is cyclic of order p^{v+1}, or*
3. *$\pi(A)$ is non-trivial and A is N-conjugate to a subgroup of $\langle \gamma \rangle.\Delta$, where γ denotes the standard p-cycle $(1 \dots p) \in \Sigma_p$.*

Further, all those of type 2 are N-conjugate.

Proof. Suppose $A \not\leqslant T$. Then we know from Lemma 3.15 that $\pi(A)$ is cyclic of order p, generated by a p-cycle, σ say. Take $a \in A$ mapping to σ. Then $a^p \in \Delta_p$ by Corollary 3.19.

If a^p is a generator of Δ_p then $a^{p^{v+1}} = 1$ and, from Corollary 3.20,

$$A \leqslant \langle a \rangle.\Delta_p = \langle a \rangle.\langle a^p \rangle = \langle a \rangle \leqslant A$$

so that $A = \langle a \rangle$ is cyclic of order p^{v+1}.

Otherwise, $a^p = \delta^p$ for some $\delta = (\delta, \dots, \delta) \in \Delta_p$. Since σ is a p-cycle, by basic combinatorics there is a permutation $\tau \in \Sigma_p$ such that $\tau\sigma\tau^{-1} = \gamma$. Put $A' = \tau A \tau^{-1}$ and $a' = \tau a \tau^{-1}$. Then we have $\pi(a') = \tau\sigma\tau^{-1} = \gamma$ and $\pi(A') = \langle \gamma \rangle$. Further, $(a')^p = \tau a^p \tau^{-1} = \tau \delta^p \tau^{-1} = \delta^p$.

Write $a' = \gamma.(b_1, \ldots, b_p)$ for some $(b_1, \ldots, b_p) \in T$. Then an application of Lemma 3.7 gives

$$(a')^p = \gamma.(b_1, \ldots, b_p) \ldots \gamma.(b_1, \ldots, b_p) = \gamma^p.(b_1 \ldots b_p, \ldots, b_1 \ldots b_p) = (b_1 \ldots b_p, \ldots, b_1 \ldots b_p)$$

so that $b_1 \ldots b_p = \delta^p$. Now, putting $u = (1, b_1\delta^{-1}, b_1b_2\delta^{-2}, \ldots, b_1 \ldots b_{p-1}\delta^{-(p-1)})$ we see that

$$\begin{aligned} u\gamma u^{-1} &= (1, b_1\delta^{-1}, \ldots, b_1 \ldots b_{p-1}\delta^{-(p-1)})\gamma(1, b_1\delta^{-1}, \ldots, b_1 \ldots b_{p-1}\delta^{-(p-1)})^{-1} \\ &= \gamma.(b_1\delta^{-1}, \ldots, b_1 \ldots b_{p-1}\delta^{-(p-1)}, 1)(1, b_1^{-1}\delta, \ldots, b_1^{-1} \ldots b_{p-1}^{-1}\delta^{p-1}) \\ &= \gamma.(b_1\delta^{-1}, \ldots, b_p\delta^{-1}) \\ &= a'\delta^{-1}. \end{aligned}$$

It follows that $u^{-1}a'u = \gamma\delta$ so that

$$(u^{-1}\tau).A.(u^{-1}\tau)^{-1} = u^{-1}A'u \leq u^{-1}(\langle a' \rangle.\Delta)u = u^{-1}\langle a' \rangle u.\Delta = \langle \gamma\delta \rangle.\Delta = \langle \gamma \rangle.\Delta$$

using Corollary 3.20. Hence A is conjugate to a subgroup of $\langle \gamma \rangle.\Delta$.

For the final statement, take $A = \langle a \rangle$ of type 2, that is cyclic of order p^{v+1}. Then we know that there is a generator $\delta \in \Delta_p$ with $a^p = \delta$.

Now, $\pi(a)$ is a p-cycle and so we can choose $\tau \in \Sigma_p$ with $\tau\pi(a)\tau^{-1} = \gamma = (1 \ldots p)$. Note that $(\tau a \tau^{-1})^p = \tau\delta\tau^{-1} = \delta$ since δ is in the centre of N. Hence A is conjugate to the group $\tau A \tau^{-1}$ which is cyclic of order p^{v+1} generated by an element of the form $\gamma(b_1, \ldots, b_p)$ with the property that $(b_1 \ldots b_p, \ldots, b_1 \ldots b_p) = \delta$.

By the above working, taking two subgroups of type 2, A and A' say, we can assume, without loss of generality, that they are generated by elements $a = \gamma(b_1, \ldots, b_p)$ and $a' = \gamma(b'_1, \ldots, b'_p)$ with $b_1 \ldots b_p = b'_1 \ldots b'_p$. Now putting $u = (b_1(b'_1)^{-1}, \ldots, b_1 \ldots b_{p-1}(b'_1 \ldots b_{p-1})^{-1}, 1)$ it is a straight forward calculation to check that

$$u.a.u^{-1} = u.\gamma(b_1 \ \ldots, b_p).u^{-1} = \gamma(b'_1, \ldots, b'_p) = a'$$

showing that A and A' are N-conjugate, as required. ☐

We are now able to give a stronger statement about the abelian p-subgroups of $GL_p(K)$. Let a_0 denote a generator of the p-part of $K^\times \simeq C_{p^v}$. As usual, we let $\gamma = \gamma_p = (1 \ldots p) \in \Sigma_p$ denote the standard p-cycle. Put $a = \gamma(a_0, 1, \ldots, 1) \in \Sigma_p \wr K^\times \subseteq GL_p(K)$ and let $A = \langle a \rangle$. Note that $a^p = (a_0, \ldots, a_0)$ so that $a^{p^{v+1}} = 1$ and A is a cyclic subgroup of $GL_p(K)$ of order p^{v+1}.

Proposition 3.22. *Let H be an abelian p-subgroup of $GL_p(K)$. Then H is $GL_p(K)$-conjugate to either a subgroup of T or to A.*

Proof. By Sylow's theorems we know that H is $GL_p(K)$-conjugate to a subgroup of $P \leqslant N$. Thus, by Proposition 3.21, it is conjugate to either a subgroup of T, A or a subgroup of $\langle \gamma \rangle.\Delta_p$. We will assume the latter case and show that H is actually $GL_p(K)$-conjugate to a subgroup of T.

We can assume, without loss of generality, that H is itself a subgroup of $\langle \gamma \rangle.\Delta$. Let $\gamma(b, \ldots, b) \in H$ where $b \in K^\times$. Then, for any $g \in GL_p(K)$, we have $g.\gamma(b, \ldots, b).g^{-1} = g\gamma g^{-1}(b, \ldots, b)$ since $(b, \ldots, b) \in Z(GL_p(K))$. Thus it remains to show that γ is diagonalisable.

Let u be a generator of $K^\times \simeq C_{p^v}$. For $k = 0, \ldots, p-1$ put

$$v_k = \begin{pmatrix} 1 \\ u^{kp^{v-1}} \\ \vdots \\ u^{k(p-1)p^{v-1}} \end{pmatrix} \in (K^\times)^p.$$

Then

$$\gamma.v_k = \begin{pmatrix} u^{kp^{v-1}} \\ \vdots \\ u^{k(p-1)p^{v-1}} \\ 1 \end{pmatrix} = u^{kp^{v-1}} \begin{pmatrix} 1 \\ u^{kp^{v-1}} \\ \vdots \\ u^{k(p-1)p^{v-1}} \end{pmatrix} = u^{kp^{v-1}} v_k.$$

Thus v_k is an eigenvector of γ with eigenvalue $u^{kp^{v-1}}$. Hence γ has distinct eigenvalues $1, u^{pv-1}, \ldots, u^{(p-1)p^{v-1}}$ and so, putting $g = (v_0|\ldots|v_{p-1})$, we find that g is invertible and

$$g^{-1}.\gamma(b, \ldots, b).g = g\gamma g^{-1}(b, \ldots, b) = (1, u^{p^{v-1}}, \ldots, u^{(p-1)p^{v-1}})(b, \ldots, b) \in T. \qquad \square$$

Later we will need some understanding of the action of $N_{GL_p(K)}(A)$ on A. We can understand this as follows.

Lemma 3.23. *Let $g \in N_{GL_p(K)}(A)$. Then, writing a for the usual generator of A, we have $gag^{-1} = a^{1+kp^v}$ for some k.*

Proof. If $g \in N_{GL_p(K)}(A)$ then $gag^{-1} = a^s$ for some s not divisible by p. But $a^p \in \Delta$ and hence $a^p = ga^pg^{-1} = (gag^{-1})^p = a^{sp} = (a^p)^s$. It follows that $p = ps \bmod p^{v+1}$ so that $p(s-1) = 0 \bmod p^{v+1}$ and therefore $s = 1 \bmod p^v$. $\square$

Chapter 4

Formal group laws and the Morava E-theories

4.1 Formal group laws

We outline the basic theory of formal group laws, covering only the material needed for the development of this thesis. For more comprehensive accounts of the area see [Frö68], [Haz78] or [Rav86]. As before, all rings and algebras are commutative and unital.

4.1.1 Basic definitions and results

Definition 4.1. Let R be a commutative ring. A *formal group law* over R is a power series $F(x,y) \in R[\![x,y]\!]$ such that

1. $F(x,0) = x$,
2. $F(x,y) = F(y,x)$,
3. $F(F(x,y),z) = F(x,F(y,z))$ in $R[\![x,y,z]\!]$.

We sometimes refer to axioms 1-3 above as identity, commutativity and associativity for F respectively.

Examples 4.2. The two easiest examples of a formal group law (defined over any ring) are the *additive formal group law*, $F_a(x,y) = x+y$, and the *multiplicative formal group law*, $F_m(x,y) = x+y+xy$. These examples are atypical, however: most formal group laws are genuine power series as opposed to polynomials.

It is perhaps not too surprising that a lot can be said about the form of such power series. We start with the following lemma. From here on F will denote a formal group law over a commutative ring R.

Lemma 4.3. *$F(x,y) = x+y$ mod (xy).*

Proof. Write $F(x,y) = \sum_{i,j} a_{ij}x^iy^j$ with $a_{ij} \in R$. Since $F(x,0) = x$ we get $a_{10} = 1$ and $a_{i0} = 0$ for $i \neq 1$. Similarly $a_{01} = 1$ and $a_{0j} = 0$ for $j \neq 1$. But modulo (xy) we have $\sum_{i,j} a_{ij}x^iy^j = a_{00} + \sum_{i>0} a_{i0}x^i + \sum_{j>0} a_{0j}x^i = x + y$, as required. □

Lemma 4.4. *(Formal inverse) For any formal group law F there exists a unique power series $\iota(x) \in R[\![x]\!]$ such that $F(x, \iota(x)) = 0$.*

Proof. We define $\iota(x)$ inductively. Put $\iota_1(x) = -x$. Then we have $F(x, \iota_1(x)) = x+(-x) = 0 \mod (x^2)$ by the previous lemma. Suppose now that we have a power series $\iota_k(x)$ such that $F(x, \iota_k(x)) = 0 \mod (x^{k+1})$ and $\iota_k(x) = 0 \mod (x)$. Write $F(x, \iota_k(x)) = ax^{k+1} \mod (x^{k+2})$ and put $\iota_{k+1}(x) = \iota_k(x) - ax^{k+1}$. Then for any $j > 0$, working modulo (x^{k+2}) we have $\iota_{k+1}(x)^j = (\iota_k(x) - ax^{k+1})^j = \iota_k(x)^j$ (since $x|\iota_k(x)$) and similarly $x^j\iota_{k+1}(x) = x^j(\iota_k(x) - ax^{k+1}) = x^j\iota_k(x)$. It follows that $F(x, \iota_{k+1}(x)) = F(x, \iota_k(x)) - ax^{k+1} = 0 \mod (x^{k+2})$.

Put $\iota(x) = \lim_{k\to\infty} \iota_k(x)$ which, since $\iota_k(x) = \iota_{k+1}(x) \mod (x^{k+1})$, is a well defined power series with $F(x, \iota(x)) = 0 \mod (x^k)$ for all k; that is $F(x, \iota(x)) = 0$. It is not hard to see that any other $f(x)$ with this property must have $f(x) = \iota_k(x) = \iota(x) \mod (x^{k+1})$ for each k so that $f(x) = \iota(x)$, proving uniqueness. □

Corollary 4.5. *With the notation of Lemma 4.4 we have $\iota(\iota(x)) = x$.*

Proof. This is an immediate consequence of uniqueness and commutativity. □

Definition 4.6. We usually write $x +_F y$ for $F(x, y)$ and refer to this as the *formal sum* of x and y. The axioms of Definition 4.1 then translate as

1. $x +_F 0 = x$,
2. $x +_F y = y +_F x$,
3. $(x +_F y) +_F z = x +_F (y +_F z)$.

Note that we may now use expressions of the form $x +_F y +_F z$ with no ambiguity.

Lemma 4.7. *$\iota(x +_F y) = \iota(x) +_F \iota(y)$.*

Proof. We have $(x+_F y)+_F(\iota(x)+_F\iota(y)) = x+_F y+_F\iota(y)+_F\iota(x) = 0$. Thus, by uniqueness, $\iota(x +_F y) = \iota(x) +_F \iota(y)$, as required. □

Definition 4.8. We sometimes write $-_Fx$ for $\iota(x)$ and define $x -_F y$ to be $F(x, \iota(y))$. For any $m \in \mathbb{N}$ we define $[m]_F(x) = x +_F \ldots +_F x$ (m times) and $[-m]_F(x) = [m](\iota(x))$. We call $[m]_F(x)$ the *m-series on x*. When there is no ambiguity we may simply write $[m](x)$ for $[m]_F(x)$.

Lemma 4.9. *For any $m \in \mathbb{Z}$ we have $[m](x +_F y) = [m](x) +_F [m](y)$.*

Proof. This follows straight from symmetry, associativity and Lemma 4.7 since we can reorder the terms in the formal sum however we like. □

Lemma 4.10. *For any $m, n \in \mathbb{Z}$ we have $[m + n](x) = [m](x) +_F [n](x)$.*

Proof. If one of m or n is zero the result follows immediately since $x +_F 0 = x$. The cases $m, n > 0$ and $m, n < 0$ are exercises in counting. Hence we can assume, without loss of generality, that $m > 0 > n$. Note that $[-n](x) = [n](\iota(x)) = \iota([n](x))$ by Lemma 4.7 so that we can take $m, n > 0$ and prove $[m-n](x) = [m](x) -_F [n](x)$.

Suppose first that $m - n \geq 0$. Then $[m-n](x) +_F [n](x) = [m](x)$ and hence $[m-n](x) = [m](x) +_F \iota([n](x)) = [m](x) -_F [n](x)$, as required. If, on the other hand, $m - n < 0$ we have $[n-m](x) = [n](x) -_F [m](x)$ by the previous workings and then

$$[m-n](x) = [n-m](\iota(x)) = \iota([n-m](x)) = \iota([n](x) -_F [m](x)) = [m](x) -_F [n](x). \qquad \square$$

Lemma 4.11. *$[m](x) = mx$ mod (x^2)*

Proof. This is a simple induction argument. We know $[1](x) = x$. If $[k](x) = kx \bmod (x^2)$ for some k then $[k+1](x) = [k](x) +_F x = [k](x) + x \bmod (x[k](x))$. It follows that, modulo (x^2), we have $[k+1](x) = [k](x) + x = kx + x = (k+1)x$. Hence $[m](x) = mx \bmod (x^2)$ for $m \geq 0$. For $m < 0$ we have $[m](x) = -_F[-m](x)$ and the result follows. □

Corollary 4.12. *$[m](x)$ is a unit multiple of x in $R[\![x]\!]$ if and only if $m \in R^\times$.*

Proof. By the previous result we have $[m](x) = x.f(x)$ for some power series $f(x)$ with constant term m. Any such series is a unit in $R[\![x]\!]$ if and only if the constant term is invertible. □

Definition 4.13. We write $\langle m \rangle(x)$ for the *divided m-series on x* which is defined to be $[m](x)/x$. Note that, by the above result, this is a unit in $R[\![x]\!]$ if and only if $m \in R^\times$.

Lemma 4.14. *$x -_F y$ is a unit multiple of $x - y$ in $R[\![x, y]\!]$.*

Proof. Using Lemma 4.3 we have $x = x -_F y +_F y = x -_F y + y + (x -_F y)yf(x,y)$ for some $f(x,y) \in R[\![x,y]\!]$. Thus we get $(x -_F y)(1 + yf(x,y)) = x - y$. Since $1 + yf(x,y)$ has invertible constant term it is a unit in $R[\![x,y]\!]$ and we are done. □

4.1.2 Formal logarithms and p-typical formal group laws

We fix a prime p and make the following definitions.

Definition 4.15. Given formal group laws F and G over a ring R, we define a homomorphism from F to G to be a power series $f(x) \in R[\![x]\!]$ with zero constant term such that $f(F(x,y)) = G(f(x), f(y))$. Such a power series is an isomorphism if and only if $f(x)$ is invertible under composition, that is if there is $g(x)$ with $f(g(x)) = x = g(f(x))$. Note that this occurs precisely when the coefficient of x is invertible in R. We call an isomorphism of formal group laws *strict* if $f(x) = x \bmod x^2$.

This construction allows us, should we so desire, to form a category of formal groups laws $FGL(R)$ over any ring R.

Proposition 4.16. *Let $F(x,y)$ be a formal group law over a $\mathbb{Q}$-algebra R. Then there is a unique power series $l_F(x) \in R[\![x]\!]$ such that $l_F(0) = 0$, ${l_F}'(0) = 1$, and $l_F(F(x,y)) = l_F(x) + l_F(y)$. That is, F is canonically isomorphic to the additive formal group law F_a.*

Proof. This is proved in [Rav86, Theorem A2.1.6]; if we write $F_2(x,y) = \partial F/\partial y$ the series $l_F(x)$ which does the job is given by

$$l_F(x) = \int_0^x \frac{dt}{F_2(t,0)}. \qquad \square$$

Definition 4.17. A strict isomorphism between a formal group law F and the additive formal group law $x+y$ is known as a *formal logarithm* for F and written $\log_F(x)$. By Proposition 4.16, such a thing exists uniquely if R is a $\mathbb{Q}$-algebra. Of course they can also occur for other rings.

Definition 4.18. Let $F(x,y)$ be a formal group law over a torsion-free $\mathbb{Z}_{(p)}$-algebra R. Then we call F *p-typical* if it has a formal logarithm of the form $\log_F(x) = x + \sum_{i>0} l_i x^{p^i}$ over $\mathbb{Q} \otimes R$.

Our next result concerns one such p-typical formal group law.

Proposition 4.19. *Let n be a positive integer. Then there is a p-typical formal group law F over $\mathbb{Z}_p[\![u_1,\ldots,u_{n-1}]\!]$ such that*

$$[p](x) = \exp_F(px) +_F u_1x^p +_F \ldots +_F u_{n-1}x^{p^{n-1}} +_F x^{p^n},$$

where $\exp_F(x)$ is the inverse to $\log_F(x)$. In particular,

$$[p](x) = u_i x^{p^i} \ mod \ (p, u_1, \ldots, u_{i-1}, x^{p^{i+1}}).$$

Proof. We give the logarithm for F explicitly, following [Str97, pp.204-205] which in turn follows [Rav86, Section 4.3]. Let $\mathcal{I}$ be the set of non-empty sequences of the form $I = (i_1,\ldots,i_m)$ with $0 < i_k \le n$ for each k. We define $|I| = m$ and $\|I\| = i_1 + \ldots + i_m$. Letting $j_k = \sum_{1\le l<k} i_l$ we put $u_I = \prod_{k=1}^m u_{i_k}^{p^{j_k}} = u_{i_1}^{p^0}.u_{i_2}^{p^{i_1}} \ldots u_{i_m}^{p^{i_1+\ldots+i_{m-1}}}$ (where we use the convention $u_n = 1$). We then let

$$l(x) = x + \sum_{I\in\mathcal{I}} \frac{u_I}{p^{|I|}} x^{p^{\|I\|}} \in \mathbb{Q}_p[\![u_1,\ldots,u_{n-1}]\!][\![x]\!]$$

and put $F(x,y) = l^{-1}(l(x)+l(y))$. It can be shown that $l(x)$ satisfies a functional equation of a suitable form so that the functional equation lemma can be applied (see [Haz78]) and F is in fact a formal group law over $\mathbb{Z}_p[\![u_1,\ldots,u_{n-1}]\!]$.

For the statement concerning the p-series, it suffices to show that

$$l([p](x)) = l(\exp_F(px) +_F u_1x^p +_F \ldots +_F u_{n-1}x^{p^{n-1}} +_F x^{p^n})$$

as the result would follow on applying l^{-1}. Now, using the notation $u_n = 1$ for simplicity, we have

$$\begin{aligned} l(\exp_F(px) +_F u_1x^p +_F \ldots +_F u_{n-1}x^{p^{n-1}} +_F x^{p^n}) &= px + l(u_1x^p) + \ldots + l(x^{p^n}) \\ &= px + \sum_{j=1}^n l(u_j x^{p^j}). \end{aligned}$$

Given $I \in \mathcal{I}$, write $I(j) = (i_1,\ldots,i_m,j)$. Then $|I(j)| = |I|+1$, $\|I(j)\| = \|I\|+j$ and

$u_{I(j)} = u_I . u_j^{p^{\|I\|}}$. Thus

$$\begin{aligned} l(u_j x^{p^j}) &= u_j x^{p^j} + \sum_{I \in \mathcal{I}} \frac{u_I}{p^{|I|}} (u_j x^{p^j})^{p^{\|I\|}} \\ &= p . \frac{u_j}{p} x^{p^j} + p \sum_{I \in \mathcal{I}} \frac{u_I}{p^{|I|+1}} u_j^{p^{\|I\|}} x^{p^{\|I\|+j}} \\ &= p \left(\frac{u_{(j)}}{p^{|(j)|}} x^{p^{\|(j)\|}} + \sum_{I \in \mathcal{I}} \frac{u_{I(j)}}{p^{|I(j)|}} x^{p^{\|I(j)\|}} \right). \end{aligned}$$

Hence

$$\begin{aligned} px + \sum_{j=1}^{n} l(u_j x^{p^j}) &= p \left(x + \sum_{j=1}^{n} \left(\frac{u_{(j)}}{p^{|(j)|}} x^{p^{\|(j)\|}} + \sum_{I \in \mathcal{I}} \frac{u_{I(j)}}{p^{|I(j)|}} x^{p^{\|I(j)\|}} \right) \right) \\ &= p \left(x + \sum_{I \in \mathcal{I}} \frac{u_I}{p^{|I|}} x^{p^{\|I\|}} \right) \\ &= p(l(x)) \\ &= l([p](x)) \end{aligned}$$

and we are done. □

Corollary 4.20. *Let n be a positive integer and let $l(x) = \sum_{i \geq 0} x^{p^{ni}}/p^i \in \mathbb{Q}_p[\![x]\!]$. Then there is a formal group law $F_1(x, y) \in \mathbb{Z}_p[\![x, y]\!]$ with $\log_{F_1}(x) = l(x)$ over $\mathbb{Q}_p$. Further, $[p](x) = l^{-1}(px) +_{F_1} x^{p^n}$.*

Proof. These claims all follow easily by reducing the results of Proposition 4.19 modulo $(u_1, \ldots, u_{n-1})$. The proof of the final statement is, perhaps, worth including as a simplified version of the proof of the corresponding result in 4.19. We have

$$\begin{aligned} l(l^{-1}(px) +_{F_1} x^{p^n}) &= l(l^{-1}(px)) + l(x^{p^n}) \\ &= px + \sum_{i \geq 0} (x^{p^n})^{p^{ni}}/p^i \\ &= px + p \sum_{i \geq 0} x^{p^{(n+1)i}}/p^{i+1} \end{aligned}$$

which is easily seen to be $pl(x) = l([p](x))$. Applying l^{-1} we get the result. □

Remark 4.21. If we define $F_0(x, y)$ to be $F_1(x, y)$ reduced mod p (a formal group law over $\mathbb{F}_p$) then it can be shown that F is the universal deformation of F_0 (see [Str97] and [LT66]). We will later use F to define our cohomology theory.

4.1.3 Formal group laws over fields of characteristic p

Lemma 4.22. *Let F be a formal group law over a field K of characteristic p. Then either $[p](x) = 0$ or there exists an integer $n > 0$ such that $[p](x) = ax^{p^n}$ mod (x^{p^n+1}) for some $a \in K^\times$.*

Proof. This is covered in [Rav86]. In fact this is a special case of a more general result, namely that any endomorphism f of F is either trivial or is such that $f(x) = g(x^{p^n})$ for some $g(x) \in R[\![x]\!]$ with $g'(0) \neq 0$ and some $n \geq 0$. Since $[p](x)$ is an endomorphism of F and

$[p](x) = px = 0$ modulo x^2 it follows that either $[p](x) = 0$ or $[p](x)$ has leading term ax^{p^n} for $a \in K^\times$ and some $n > 0$. $\square$

Given a formal group law F over K of characteristic p we define the *height* of F to be the integer n occurring in Lemma 4.22 or ∞ if $[p](x) = 0$. This is an isomorphism invariant (that is, isomorphic formal group laws have the same height). Further, for any field K of characteristic p there exists a formal group law of height n for each $n > 0$ (see [Rav86]).

Given a complete local $\mathbb{Z}_p$-algebra $(R, \mathfrak{m})$ (that is, a complete local ring with $p \in \mathfrak{m}$) and a formal group law F over R we can reduce the coefficients of F modulo $\mathfrak{m}$ to get a formal group law F_0 over $R/\mathfrak{m}$ which is a field of characteristic p. We then define the height of F to be the height of its mod-$\mathfrak{m}$ reduction F_0.

4.1.4 Lazard's ring and the universal formal group law

Definition 4.23. Given a ring homomorphism $\phi : R \to S$ and a formal group law F over R we obtain a new formal group law ϕF over S by applying ϕ to the coefficients of F. Note that if $\log_F$ exists then, by uniqueness, $\log_{\phi F}$ exists and is equal to $\phi \log_F$.

The following is a result of Lazard.

Proposition 4.24. *There is a ring L and a formal group law F_{univ} over L such that for any ring R and any formal group law F over R there is a unique homomorphism $\phi : L \to R$ such that $\phi F_{univ} = F$.*

Proof. Let S be the polynomial ring over $\mathbb{Z}$ generated by symbols $\{a_{i,j} \mid i, j \geq 0\}$. Let G be the power series $G(x, y) = \sum_{i,j} a_{i,j} x^i y^j \in S[\![x, y]\!]$. Then, letting I be the ideal in S generated by the relations that would force G to be a formal group law, on passing to the quotient ring $L = S/I$ we get a formal group law F_{univ} over S/I. Given any formal group law F over R there is a unique map $S \to R$ sending $a_{i,j}$ to the coefficient of $x^i y^j$ in $F(x, y)$. It is clear this map factors through a map $\phi : S/I \to R$ which has the properties claimed. $\square$

The ring L is often referred to as *Lazard's ring* and F_{univ} the *universal formal group law*, for obvious reasons. We will see later that L has a fundamental role in the development of a certain class of cohomology theories, of which the Morava E-theories are examples.

4.1.5 The Weierstrass preparation theorem

Let $(R, \mathfrak{m})$ be a complete local ring. Then $f(x) = \sum_i a_i x^i \in R[\![x]\!]$ is a *Weierstrass series of degree d* if $a_0, \ldots, a_{d-1} \in \mathfrak{m}$ and $a_d \in R^\times$. We call f a *Weierstrass polynomial (of degree d)* if, in addition, $a_i = 0$ for all $i > d$, that is if f is in fact a polynomial of degree d. We have the following theorem.

Lemma 4.25. *(Weierstrass preparation theorem) Let $(R, \mathfrak{m})$ be a (graded) complete local ring. If $f(x) \in R[\![x]\!]$ is a Weierstrass series of degree d there is a unique factorisation $f(x) = u(x)g(x)$ where $g(x)$ is a Weierstrass polynomial of degree d and $u(x)$ is a unit in $R[\![x]\!]$.*

Proof. This is proved in [Lan78]. $\square$

Corollary 4.26. *If $f(x) \in R[\![x]\!]$ is a Weierstrass polynomial of degree d then $R[\![x]\!]/(f(x))$ is a free R-module of rank d with basis $\{1, x, \ldots, x^{d-1}\}$.*

Proof. By the Weierstrass preparation theorem $f(x)$ is a unit multiple of a monic polynomial of degree d and the result follows. □

We see the relevance of this diversion in the following result.

Proposition 4.27. *Let F be a formal group law of height n over a (graded) complete local $\mathbb{Z}_p$-algebra $(R, \mathfrak{m})$. Then, for each r, $[p^r](x)$ is a Weierstrass series of degree p^{nr}.*

Proof. Since F has height n we know that $[p](x) = ax^{p^n}$ modulo $\mathfrak{m}, x^{p^n+1}$ for some $a \in (R/\mathfrak{m})^\times$. Thus, writing $[p](x) = \sum_i a_i x^i$, we have $a_0, \ldots, a_{p^n-1} \in \mathfrak{m}$ and, since a_{p^n} is invertible modulo $\mathfrak{m}$, there is $b \in R$ with $a_{p^n} b = 1 - m$ for some $m \in \mathfrak{m}$. Then $a_{p^n} b(1 + m + m^2 + \ldots) = (1 - m)(1 + m + m^2 + \ldots) = 1$ so that $a_{p^n} \in R^\times$ and $[p](x)$ is a Weierstrass series of degree p^n. Using the fact that $[p^{i+1}](x) = [p]([p^i](x))$ it then follows easily that $[p^r](x)$ is a Weierstrass series of degree p^{nr}. □

Corollary 4.28. *Let F and R be as above. Then $R[\![x]\!]/([p^r](x))$ is free over R of rank p^{nr} with basis $\{1, x, \ldots, x^{p^{nr}-1}\}$.*

Definition 4.29. Let F be the formal group law over $\mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!]$ of Proposition 4.19. Then, using the Weierstrass preparation theorem, we define $g_r(t)$ to be the Weierstrass polynomial of degree p^{nr} which is a unit multiple of $[p^r]_F(t)$ in $\mathbb{Z}_p[\![u_1, \ldots, u_n]\!][t]$.

4.1.6 Formal group laws over complete local $\mathbb{Z}_p$-algebras

Lemma 4.30. *Let $(R, \mathfrak{m})$ be a complete local $\mathbb{Z}_p$-algebra and let F be a formal group law over R. Then, writing $\mathfrak{m}_{R[\![x]\!]}$ for the maximal ideal of $R[\![x]\!]$, we have $[p^r](x) \in (\mathfrak{m}_{R[\![x]\!]})^r$.*

Proof. Working modulo $\mathfrak{m}$ we find that either $[p](x) = 0$ or $[p](x) = ax^{p^n} \bmod (x^{p^n+1})$ for some n; in either case we conclude that $[p](x) \in \mathfrak{m}_{R[\![x]\!]}$. Noting that the $[p](0) = 0$ a simple induction argument shows that $[p^{r+1}](x) = [p]([p^r](x)) \in (\mathfrak{m}_{R[\![x]\!]})^{r+1}$, as required. □

This gives us the following useful result.

Lemma 4.31. *Let R be a $\mathbb{Z}_p$-algebra and F a formal group law over R. Then given any $a \in \mathbb{Z}_p$ there is a well-defined power series $[a](x) \in R[\![x]\!]$ such that*

1. *if $a \in \mathbb{Z}$ then $[a](x)$ coincides with the standard a-series on x,*
2. *$[a]([b](x)) = [a.b](x)$, and*
3. *if (a_i) is a sequence in $\mathbb{Z}_p$ converging to a then $[a_i](x)$ converges to $[a](x)$ in $R[\![x]\!]$.*

Proof. Take $a \in \mathbb{Z}_p$ and write $a = \sum_{i=0}^\infty a_i p^i$ and put $\alpha_k = \sum_{i=0}^k a_i p^i$. Then, using Lemma 4.14, we have

$$[\alpha_{k+1}](x) - [\alpha_k](x) \sim [\alpha_{k+1}](x) -_F [\alpha_k](x) = \left[a_{k+1} p^{k+1}\right](x) = [a_{k+1}]([p^{k+1}](x)),$$

where $a \sim b$ denotes that a is a unit multiple of b. Hence $[\alpha_{k+1}](x) - [\alpha_k](x) \in (\mathfrak{m}_{R[\![x]\!]})^{k+1}$ using Lemma 4.30 and the limit

$$[a](x) = \left[\sum_{i=0}^{\infty} a_i p^i\right](x) = \lim_{k\to\infty} [\alpha_k](x)$$

is well defined. It is straightforward to check that this definition of $[a](x)$ satisfies the properties listed. $\square$

Lemma 4.32. *Let R be a torsion-free $\mathbb{Z}_p$-algebra and F a formal group law over R. Then, for any $a \in \mathbb{Z}_p$, we have $\log_F([a](x)) = a\log_F(x)$.*

Proof. First note that, for any x and y,

$$\log_F(x) = (\log_F(x) - \log_F(y)) + \log_F(y) = \log_F(\log_F^{-1}(\log_F(x) - \log_F(y)) +_F y).$$

Hence $x = \log_F^{-1}(\log_F(x) - \log_F(y)) +_F y$ so that $\log_F(x -_F y) = \log_F(x) - \log_F(y)$. Then for $a \in \mathbb{Z}_p$, using the notation of the proof of Lemma 4.31, we have

$$\log_F([a](x)) - \log_F([\alpha_k](x)) = \log_F([a](x) -_F [\alpha_k](x)) = \log_F([\textstyle\sum_{i=k+1}^{\infty} a_i p^i](x))$$

which lies in $(\mathfrak{m}_{R[\![x]\!]})^{k+1}$, so that

$$\log_F([a](x)) = \lim_{k\to\infty} \log_F([\alpha_k](x)) = \lim_{k\to\infty} \alpha_k \log_F(x) = a\log_F(x). \qquad \square$$

Lemma 4.33. *Let R be a torsion-free $\mathbb{Z}_p$-algebra and F a p-typical formal group law over R. Then, for all $k \in (\mathbb{Z}/p)^\times$, we have $[\hat{k}](x) = \hat{k}x$, where $\hat{k}$ denotes the Teichmüller lift of k of section 2.1.3.*

Proof. Recall that, by definition, F has a logarithm over $\mathbb{Q} \otimes R$ of the form $\log_F(x) = x + \sum_{i>0} l_i x^{p^i}$. Let $k \in (\mathbb{Z}/p)^\times$. Then, since $\hat{k}^{p-1} = 1$ we have $\hat{k}^{p^i} = \hat{k}$ for all $i > 0$. Hence

$$\begin{aligned}
\log_F([\hat{k}](x)) &= \hat{k}\log_F(x) \\
&= \hat{k}x + \sum_{i>0} l_i \hat{k} x^{p^i} \\
&= \hat{k}x + \sum_{i>0} l_i (\hat{k}x)^{p^i} \\
&= \log_F(\hat{k}x)
\end{aligned}$$

The result follows on applying $\log_F^{-1}$ to both sides. $\square$

Corollary 4.34. *Let R be a torsion-free $\mathbb{Z}_p$-algebra and F a p-typical formal group law over R. Then $\langle p\rangle([\hat{k}](x)) = \langle p\rangle(x)$ for all $k \in (\mathbb{Z}/p)^\times$.*

Proof. By Lemma 4.33 we have $[p]([\hat{k}](x)) = [p\hat{k}](x) = [\hat{k}]([p](x)) = \hat{k}[p](x) = \hat{k}x\langle p\rangle(x)$. But $[p]([\hat{k}](x)) = [\hat{k}](x)\langle p\rangle([\hat{k}](x)) = \hat{k}x\langle p\rangle([\hat{k}](x))$ and so, since $\hat{k}$ is a unit and x is not a zero divisor in $R[\![x]\!]$, we get $\langle p\rangle([\hat{k}](x)) = \langle p\rangle(x)$. $\square$

4.2 The Morava E-theories

We outline the development of the cohomology theories that we will be using. There is some variation in the literature, but this definition is consistent with relevant earlier work of Strickland and others. Full accounts of this material are not easy to come by, but good starting points are [Rav92] and [Rav86]. A thorough treatment of related theory is found in [HS99].

4.2.1 Complex oriented cohomology theories

A *(multiplicative) cohomology theory* is a contravariant functor from topological spaces to graded rings satisfying the first three of the Eilenberg-Steenrod axioms. More precisely, we make the following definitions.

Definition 4.35. We define the category of *CW pairs* to be the category with objects (X, A), where A is a subcomplex of the CW complex X, and morphisms $(X, A) \to (Y, B)$ given by continuous cellular maps $X \to Y$ which restrict to a map $A \to B$. We sometimes write X for the object $(X, \emptyset)$.

A *generalised cohomology theory*, h, is a contravariant functor from CW pairs to $\mathbb{Z}$-graded abelian groups satisfying the following conditions.

- If $f, g : (X, A) \to (Y, B)$ are homotopic then $f^* = g^* : h^*(Y, B) \to h^*(X, A)$.
- Writing $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ there are *connecting homomorphisms* $\partial^q : h^q(A) \to h^{q+1}(X, A)$ for each q such that there is a natural long exact sequence

$$\ldots \xrightarrow{\partial^{q-1}} h^q(X, A) \xrightarrow{h^q(j)} h^q(X) \xrightarrow{h^q(i)} h^q(A) \xrightarrow{\partial^q} h^{q+1}(X, A) \xrightarrow{h^{q+1}(j)} \ldots$$

- If U is an open subset of X with $\overline{U}$ contained in the interior of A and such that $(X \setminus U, A \setminus U)$ can be given a CW structure, then the map $j : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$h^*(j) : h^*(X, A) \xrightarrow{\sim} h^*(X \setminus U, A \setminus U).$$

An immediate consequence of the definition is that if X is homotopy equivalent to Y then $h^*(X) \simeq h^*(Y)$. We define the *coefficients* of the cohomology theory to be the graded abelian group $h^* = h^*(pt)$, where pt is the one-point space. Note that for any space X there is a unique map $X \to pt$ giving a map $h^*(pt) \to h^*(X)$ which makes $h^*(X)$ a module over h^*.

We define a functor $\tilde{h}$ from topological spaces to graded abelian groups known as the *reduced theory* by $\tilde{h}^*(X) = \operatorname{coker}(h^*(pt) \to h^*(X))$. Note that, by choosing a map $pt \to X$, we get a splitting $h^*(X) \simeq \tilde{h}^*(X) \oplus h^*(pt)$.

Often a cohomology theory will have additional structure making the groups $h^*(X, A)$ into graded rings, commutative in the graded sense so that if $a \in h^i(X)$ and $b \in h^j(X)$ then $ab = (-1)^{ij} ba$. In such a situation we say that h is *multiplicative.*

We call a cohomology theory h *complex oriented* if there is a class $x \in h^2(\mathbb{C}P^\infty)$ such that its restriction to $h^2(\mathbb{C}P^1)$ generates $\tilde{h}^2(\mathbb{C}P^1)$ as an h^0-module (see [Ada74]). The class x is known as a *complex orientation* for h.

Any complex oriented cohomology theory h with complex orientation x satisfies $h^*(\mathbb{C}P^\infty) = h^*[\![x]\!]$ and $h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = h^*[\![\pi_1^*(x), \pi_2^*(x)]\!]$ where π_1 and π_2 are the two projection maps $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ (again, see [Ada74]). Since $\mathbb{C}P^\infty = BS^1$, the commutative multiplication map $S^1 \times S^1 \to S^1$ gives a product $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ making $\mathbb{C}P^\infty$ into an H-space. The induced map $\mu^* : h^*(\mathbb{C}P^\infty) \to h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ sends the complex orientation x to a power series $F(\pi_1^*(x), \pi_2^*(x)) = F(x_1, x_2)$.

Lemma 4.36. *The power series $F(x_1, x_2) = \mu^*(x)$ is a formal group law over h^*.*

Proof. We check that the axioms for a formal group law hold. Firstly, write $j : S^1 \to S^1 \times S^1$ for the map $z \mapsto (z, 1)$. Then the commutative diagram

$$\begin{tikzcd} S^1 \arrow[d, "j"'] \arrow[dr, "\mathrm{id}_{S^1}"] & \\ S^1 \times S^1 \arrow[r, "\mu"'] & S^1 \end{tikzcd} \qquad \text{induces} \qquad \begin{tikzcd} h^*(\mathbb{C}P^\infty) \arrow[dr, equal] & \\ h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \arrow[u, "j^*"] & h^*(\mathbb{C}P^\infty) \arrow[l, "\mu^*"] \end{tikzcd}$$

and one can check that $j^*(x_1) = x$ and $j^*(x_2) = 0$ so that $F(x, 0) = j^*(F(x_1, x_2)) = j^*(\mu^*(x)) = x$. It is easy to see that, writing $\tau : S^1 \times S^1 \to S^1 \times S^1$ for the twist map, we have $\mu \circ \tau = \mu$ and that this, on passing to cohomology, gives $F(x_1, x_2) = F(x_2, x_1)$. The final axiom is a consequence of the associativity diagram

$$\begin{tikzcd} S^1 \times S^1 \times S^1 \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & S^1 \times S^1 \arrow[d, "\mu"] \\ S^1 \times S^1 \arrow[r, "\mu"'] & S^1. \end{tikzcd}$$

□

4.2.2 Defining the Morava E-theories

We aim to define our cohomology theory E and, in doing so, fix a complex orientation with favourable properties.

Lemma 4.37. *Let h be a cohomology theory such that h^* is concentrated in even degrees. Then there exists $y \in h^2(\mathbb{C}P^\infty)$ such that, for each $n > 0$, $h^*(\mathbb{C}P^n) = h^*[\![y]\!]/y^n$ and $h^*(\mathbb{C}P^\infty) = h^*[\![y]\!]$. In particular h is complex oriented.*

Proof. This is an application of the Atiyah-Hirzebruch spectral sequence

$$H^*(\mathbb{C}P^n; h^*) \Rightarrow h^*(\mathbb{C}P^n)$$

where H denotes ordinary (singular) cohomology (see [Ada74] for further details). By consideration of the cellular structure of $\mathbb{C}P^n$ we have

$$H^k(\mathbb{C}P^n; A) = \begin{cases} A & \text{for } k = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

which, in particular, lies in even degrees. Since h^* is also concentrated in even degrees it follows that all terms with at least one degree odd in the E_2-page are zero and the spectral sequence collapses. As usual, writing $J_k = \ker(h^*(\mathbb{C}P^n) \to h^*(\mathrm{skel}^{2k}(\mathbb{C}P^n)))$ we have a canonical isomorphism $J_k/J_{k+1} \simeq H^{2k}(\mathbb{C}P^n; h^*)$ so that, in particular, $J_1/J_2 \simeq h^*.x$, where x is the chern class of the tautological line bundle over $\mathbb{C}P^n$. Lifting x under this map gives a homogeneous element $y_n \in J_1 \subset h^*(\mathbb{C}P^n)$ of degree 2 such that $h^*(\mathbb{C}P^n) = h^*[\![y_n]\!]/y_n^n$. By naturality, we can make sure the elements $y_i \in h^2(\mathbb{C}P^i)$ are compatible for each i and hence we get $y \in \varprojlim h^2(\mathbb{C}P^n)$. Since the maps $h^*(\mathbb{C}P^n) \to h^*(\mathbb{C}P^{n-1})$ are all surjective, an application of the Milnor-sequence (again, see [Ada74]) gives $h^*(\mathbb{C}P^\infty) = \varprojlim h^*(\mathbb{C}P^n) = h^*[\![y]\!]$ and y is a complex orientation for h. □

We turn our attention to complex cobordism and have the following well known result.

Lemma 4.38. *The complex cobordism spectrum MU is complex oriented and there is a canonical orientation $x_{MU} \in MU^2(\mathbb{C}P^\infty)$.*

Proof. For more details see, for example, [Rav86, Chapter 4]. It is known that $MU^* = \mathbb{Z}[a_1, a_2, \ldots]$ with $|a_i| = -2i$, so that MU^* is concentrated in even degrees and Lemma 4.37 applies. A canonical orientation is the class corresponding to the map

$$\mathbb{C}P^\infty = BU(1) \simeq MU(1) \longrightarrow \Sigma^2 MU. \qquad \square$$

Now, using the coordinate x_{MU} of Lemma 4.38 we get an identification $MU^*(\mathbb{C}P^\infty) = MU^*[\![x_{MU}]\!]$ and, as usual, we use the multiplication map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ to get formal group law

$$F_{MU}(x_1, x_2) = \mu^*(x_{MU}) \in MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = MU^*[\![x_1, x_2]\!].$$

This is classified by a map $L \to MU^*$ where L is the Lazard ring and we have the following famous theorem of Quillen.

Proposition 4.39 (Quillen's theorem)**.** *The map $L \to MU^*$ classifying F_{MU} is an isomorphism.*

Proof. This is the main result of [Qui69] and is covered in [Rav92]. $\square$

For each prime p and $k \geq 0$, let $v_{p,k} \in MU^*$ be the coefficient of x^{p^k} in the p-series for F_{MU} and let $I_{p,k} = (v_{p,0}, v_{p,1}, \ldots, v_{p,k-1}) \lhd MU^*$. Note that $v_{p,0} = p$ and $I_{p,0}$ is defined to be 0. We use the following result of Landweber.

Proposition 4.40 (Exact functor theorem)**.** *Let M be an MU^*-module. Then the functor $X \mapsto M \otimes_{MU^*} MU_*(X)$ defines a homology theory if and only if for each prime p and each $k \geq 0$ multiplication by $v_{p,k}$ in $M/I_{p,k}M$ is injective. In particular, there is a spectrum E with $E_*(X) = M \otimes_{MU^*} MU_*(X)$.*

Proof. See [Lan76] and [HS99]. $\square$

Now, fixing a prime p and an integer $n > 0$, let $R = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}]$ and let F be the p-typical formal group law over $\mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!]$ of Proposition 4.19. Define a $\mathbb{Z}_p$-algebra map $\phi : \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!] \to R$ by $u_i \mapsto u^{p^i - 1}.u_i$ and let ϕF be the formal group law obtained by applying ϕ to the coefficients of F. We give R a grading by letting each u_i lie in degree 0 and u lie in degree -2. Then, using Quillen's theorem, ϕF is classified by a map $MU^* \to R$ which respects the grading. We show that, equipped with this map, the MU^*-module R satisfies the the exact functor theorem. Recall that we have

$$[p]_F(x) = \exp_F(px) +_F u_1 x^p +_F \ldots +_F u_{n-1} x^{p^{n-1}} +_F x^{p^n} \in \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][\![x]\!]$$

so that

$$[p]_{\phi F}(y) = \exp_{\phi F}(py) +_{\phi F} u^{p-1} u_1 y^p +_{\phi F} \ldots +_{\phi F} u^{p^{n-1}-1} u_{n-1} y^{p^{n-1}} +_{\phi F} y^{p^n} \in R[\![y]\!].$$

We use the following lemma.

Lemma 4.41. *For any $1 \le k < n$ we have $R/I_{p,k}R = \mathbb{F}_p[\![u_k, \dots, u_{n-1}]\!][u, u^{-1}]$ and $v_{p,k}$ acts as multiplication by $u^{p^k-1}u_k$. Further $R/I_{p,n}R = \mathbb{F}_p[u, u^{-1}]$ and $v_{p,n}$ acts as the identity map.*

Proof. We proceed by induction on k. For $k = 1$ we have

$$R/I_{p,k}R = R/pR = \mathbb{F}_p[\![u_1, \dots, u_{n-1}]\!][u, u^{-1}]$$

and $[p]_{\phi F}(y) = u^{p-1}u_1 y^p +_{\phi F} \dots +_{\phi F} y^{p^n}$. It follows that the coefficient of y^p in $[p]_{\phi F}(y)$ is $u^{p-1}u_1$, so that $v_{p,1} \mapsto u^{p-1}u_1$ in $R/I_{p,1}R$. The induction step is similar, noting that $v_{p,k}$ acts as a unit multiple of u_k in $R/I_{p,k}R$ we have $R/I_{p,k+1}R = (R/I_{p,k}R)/v_{p,k}(R/I_{p,k}R) = \mathbb{F}_p[\![u_{k+1}, \dots, u_{n-1}]\!][u, u^{-1}]$. □

Thus we get the following corollary.

Proposition 4.42. *The MU^*-module R satisfies the conditions of the exact functor theorem and hence there is a spectrum E with $E_*(X) = R \otimes_{MU^*} MU_*(X)$.*

Proof. The cases with $k = 0$ are immediate since R is torsion-free. The cases at the prime p with $1 \le k \le n$ are covered by Lemma 4.41 since multiplication by u_k is injective. For $k > n$ we have $R/I_{p,k}R = 0$ since $v_{p,n} \in I_{p,k}$ and hence $1 \in I_{p,k}R$. If q is a prime different to p then q is invertible in R and $q \in I_{q,k}$ for all $k \ge 1$, so that $R/I_{q,k}R = 0$ and, again, there is nothing to check. Hence the conditions of the Exact Functor Theorem hold. □

As usual, we can now use the spectrum E to define a cohomology theory. This has the following properties.

Proposition 4.43. *The cohomology theory E outlined above is multiplicative and complex oriented and there is a canonical map $\theta_X : MU^*(X) \to E^*(X)$ for each X. In particular, the map $MU^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty)$ sends the complex orientation x_{MU} to an orientation $x = \theta(x_{MU})$ which gives rise to the formal group law ϕF.*

Proof. Since $E^* = R$ is concentrated in even degrees we see that E is complex oriented by Lemma 4.37. That E is multiplicative is covered in [HS99, Proposition 2.21] and using [HS99, Proposition 2.20] we get a map of spectra $\theta : MU \to E$ which induces the map $\theta_X : MU^*(X) \to E^*(X)$ for each X. It remains to show that $\theta(x_{MU}) \in E^2(\mathbb{C}P^\infty)$ is a complex orientation for E.

By naturality of the Atiyah-Hirzebruch spectral sequence, there is a commutative diagram

$$\begin{tikzcd} H^*(\mathbb{C}P^\infty; MU^*) \arrow[r, Rightarrow] \arrow[d] & MU^*(\mathbb{C}P^\infty) \arrow[d] \\ H^*(\mathbb{C}P^\infty; E^*) \arrow[r, Rightarrow] & E^*(\mathbb{C}P^\infty) \end{tikzcd} .$$

By the same arguments as in the proof of Lemma 4.37 the map of the E_∞-pages just corresponds to the map $MU^*[\![x]\!] \to E^*[\![x]\!]$, $\sum_i a_i x^i \mapsto \sum_i \theta(a_i)x^i$. Since x_{MU} is a lift of the class x to $MU^2(\mathbb{C}P^\infty)$ it follows that $\theta(x_{MU})$ is a lift of x to $E^2(\mathbb{C}P^\infty)$ and hence is an orientation for E. By naturality we then have a commutative square

$$\begin{tikzcd} MU^*(\mathbb{C}P^\infty) \arrow[r, "\mu^*"] \arrow[d, "\theta"'] & MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \arrow[d, "\theta"] \\ E^*(\mathbb{C}P^\infty) \arrow[r, "\mu^*"'] & E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \end{tikzcd}$$

and it follows that $\mu^*(\theta(x_{MU})) = \theta(\mu^*(x_{MU})) = \theta(F(x_1, x_2)) = (\phi F)(\theta(x_1), \theta(x_2))$, showing that the formal group law associated to $\theta(x_{MU})$ is ϕF, as required. □

Corollary 4.44. *Let $y = \theta(x_{MU})$ be the complex orientation for E^* defined above and put $x = u.y \in E^0(\mathbb{C}P^\infty)$. Then $E^0(\mathbb{C}P^\infty) = E^0[\![x]\!]$ and*

$$\mu^*(x) = F(x_1, x_2) \in E^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^0[\![x_1, x_2]\!],$$

where F is the standard p-typical formal group law of Proposition 4.19.

Proof. For the first statement, recall that $E^*(\mathbb{C}P^\infty) = E^*[\![y]\!] = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}][\![y]\!]$ with $|u| = -2$ and $|y| = 2$. First note that $E^0[\![x]\!]$ is clearly contained in $E^0(\mathbb{C}P^\infty)$ since x has degree zero. Now, take $a \in E^0(\mathbb{C}P^\infty)$ and write $a = \sum_i a_i y^i$, where $a_i \in E^*$ for each i. Then $|a_i| = -2i$ so that we have $a_i = u^i a_i'$ for some $a_i' \in E^0$. Hence $a = \sum_i a_i'(uy)^i = \sum_i a_i' x^i \in E^0[\![x]\!]$.

For the second statement, we have

$$\mu^*(x) = \mu^*(uy) = u\mu^*(y) = u(\phi F)(y_1, y_2) = u(\phi F)(u^{-1}x_1, u^{-1}x_2).$$

Note that, by uniqueness of the logarithm, we have $\log_{\phi F}(t) = (\phi \log_F)(t)$. Further

$$\log_{\phi F}(u^{-1}t) = (\phi \log_F)(u^{-1}t) = u^{-1}t + \sum_I \frac{u_I}{p^{|I|}}(u^{-1}t)^{p^{\|I\|}}.u^N$$

where $u^N = (u^{p^{i_1}-1}).(u^{p^{i_2}-1})^{p^{i_1}} \ldots (u^{p^{i_m}-1})^{p^{i_1+\ldots+i_{m-1}}} = u^{p^{\|I\|}-1}$ is the factor coming from the application of ϕ to the coefficients. Hence $\log_{\phi F}(u^{-1}t) = u^{-1}\log_F(t)$. Further, $u^{-1}t = \log_{\phi F}^{-1}(u^{-1}\log_F(t))$ so that $u^{-1}\log_F^{-1}(s) = \log_{\phi F}^{-1}(u^{-1}s)$. Hence

$$\begin{aligned} u(\phi F)(u^{-1}x_1, u^{-1}x_2) &= u\log_{\phi F}^{-1}(\log_{\phi F}(u^{-1}x_1) + \log_{\phi F}(u^{-1}x_2)) \\ &= u\log_{\phi F}^{-1}(u^{-1}\log_F(x_1) + u^{-1}\log_F(x_2)) \\ &= uu^{-1}\log_F^{-1}(\log_F(x_1) + \log_F(x_2)) \\ &= F(x_1, x_2) \end{aligned}$$

so that $\mu^*(x) = F(x_1, x_2)$, as claimed. □

Definition 4.45. We refer to the theory E developed above as the *Morava E-theory of height n at the prime p*. Clearly there is one such theory for each choice of prime p and each integer $n > 0$. Note that the coefficient ring $E^* = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}]$ is concentrated in even degrees and there is an invertible element, u in degree -2. It follows that multiplication by u gives rise to an isomorphism $E^{k+2}(X) \xrightarrow{\sim} E^k(X)$ for all X and all k. We refer to the class $x = \theta(x_{MU}) \in E^2(\mathbb{C}P^\infty)$ as the *standard complex orientation for E* and the class $u.x \in E^0(\mathbb{C}P^\infty)$ as the *standard complex coordinate for E*. Often, when working in degree 0, we will abuse notation slightly and write the latter simply as x.

Remark 4.46. Using a modified exact functor theorem due to Yagita ([Yag76]) one can define, for each prime p and each $n > 0$, a related cohomology theory K with $K^* = \mathbb{F}_p[u, u^{-1}]$ where $u \in K^{-2}$. We refer to this as the *Morava K-theory of height n at p*. The convention here is slightly non-standard: in the literature, the term Morava K-theory is often used with reference to a theory $K(n)$ with $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$ where $v_n \in K(n)^{-2(p^n-1)}$. In fact K is just a modified version of $K(n)$ obtained by setting $K^*(X) = \mathbb{F}_p[u, u^{-1}] \otimes_{K(n)^*} K(n)^*(X)$, where $\mathbb{F}_p[u, u^{-1}]$ is made into a $K(n)^*$-module by letting v_n act as u^{p^n-1}. See [Rav92] for further details on these theories.

4.2.3 The cohomology of finite abelian groups

Lemma 4.47. *Let C_m be the cyclic subgroup of S^1 of order m. Then, writing x for the restriction of the complex orientation $x \in E^*(\mathbb{C}P^\infty) = E^*(BS^1)$ to $E^*(BC_m)$, we have $E^*(BC_m) = E^*[\![x]\!]/([m](x))$.*

Proof. This is Lemma 5.7 in [HKR00]. $\square$

Corollary 4.48. *Let $m = ap^r$ where a is coprime to p. Then C_{p^r} is a subgroup of C_m and the restriction map $E^*(BC_m) \to E^*(BC_{p^r})$ is an isomorphism.*

Proof. Since $[m](x) = [ap^r](x) = [a]([p^r](x))$ and $[a](x)$ is a unit multiple of x by Corollary 4.12 we see that $[m](x)$ is a unit multiple of $[p^r](x)$. Hence

$$E^*(BC_m) = E^*[\![x]\!]/([m](x)) = E^*[\![x]\!]/([p^r](x)) \xrightarrow{\sim} E^*(BC_{p^r}). \qquad \square$$

Proposition 4.49 (Künneth isomorphism)**.** *Let X be any space and Y be a space with $E^*(Y)$ free and finitely generated over E^*. Then the map $E^*(X) \otimes_{E^*} E^*(Y) \to E^*(X \times Y)$ is an isomorphism.*

Proof. This is Lemma 5.9 in [HKR00]. $\square$

Corollary 4.50. *If G is any group then, for any $m > 0$,*

$$E^*(B(G \times C_m)) \simeq E^*(BG) \otimes_{E^*} E^*(BC_m).$$

Proof. As in 4.48 we have $E^*(BC_m) \simeq E^*(BC_{p^r}) = E^0[\![x]\!]/([p^r](x))$ for some r and the latter is finitely generated and free over E^* by the Weierstrass preparation theorem. Hence the Künneth isomorphism holds. $\square$

We are now able to compute the Morava E-theory of any finite abelian group A by writing A as a product of cyclic groups and applying Corollary 4.50 repeatedly. That is, we have the following.

Proposition 4.51. *Let A be a finite abelian group with $A \simeq \prod_i C_{m_i}$. Then there is an isomorphism*

$$E^*(BA) \simeq E^*[\![x_1, \ldots, x_r]\!]/([m_1](x_1), \ldots, [m_r](x_r))$$

where, writing α_i for the map $A \twoheadrightarrow C_{m_i} \rightarrowtail S^1$, we have $x_i = \alpha_i^*(x)$.

Corollary 4.52. *Let A be a finite abelian group and let $A_{(p)} = \{a \in A \mid a^{p^s} = 1$ for some $s\}$ be the p-part of A. Then the restriction map $E^*(BA) \to E^*(BA_{(p)})$ is an isomorphism.*

Proof. We write A as a product of cyclic groups, say $A \simeq \prod_i C_{m_i}$ where $m_i = a_i p^{r_i}$. Then, using the Künneth isomorphism, we have

$$\begin{tikzcd} E^*(BA) \arrow[r, "\sim"] \arrow[d] & \bigotimes_{E^*} E^*(BC_{m_i}) \arrow[d] \\ E^*(BA_{(p)}) \arrow[r, "\sim"] & \bigotimes_{E^*} E^*(BC_{p^{r_i}}). \end{tikzcd}$$

By Corollary 4.48 the right hand map is an isomorphism and hence so is the left-hand one. $\square$

4.3 The cohomology of classifying spaces

We outline some general theory that will be used in proving our results.

Proposition 4.53. *If G is a finite group then $E^*(BG)$ is finitely generated as an E^*-module.*

Proof. This is Corollary 4.4 in [GS99], although the related proof that $K(n)^*(BG)$ is finitely generated goes back to Ravenel [Rav82]. Note that for a G-space Z they define $E^*_G(Z) = E^*(EG \times_G Z)$ and letting Z be a single point gives $E^*_G(Z) = E^*(BG)$. $\square$

Proposition 4.54. *Suppose X is a space with $E^*(X)$ finitely generated over E^* and with $K(n)^*(X)$ concentrated in even degrees. Then $E^*(X)$ is free over E^* and concentrated in even degrees.*

Proof. This is Proposition 3.5 from [Str98]. $\square$

Recall that $K^* = \mathbb{F}_p[u, u^{-1}]$. This is a module over $E^* = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}]$ under the map sending $u_i \mapsto 0$ for $i = 0, \ldots, n-1$ (with u_0 understood to be p). We find that we can often recover $K^*(BG)$ from $E^*(BG)$.

Proposition 4.55. *If $E^*(BG)$ is free over E^* then $K^*(BG) = K^* \otimes_{E^*} E^*(BG)$.*

Proof. This is Corollary 3.8 in [Str98]. $\square$

We assemble the above results to arrive at the following.

Proposition 4.56. *Let G be a finite group with $K(n)^*(BG)$ concentrated in even degrees. Then $E^*(BG)$ is free over E^* and concentrated in even degrees. Further*

$$E^i(BG) \simeq \begin{cases} E^0(BG) & \text{if } i \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad K^i(BG) \simeq \begin{cases} K^0 \otimes_{E^0} E^0(BG) & \text{if } i \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first statement follows straight from Propositions 4.54 and 4.53. Since E^* contains the unit $u \in E^{-2}$, multiplication by u^{-i} provides an isomorphism $E^0(BG) \xrightarrow{\sim} E^{2i}(BG)$, proving the statements about $E^i(BG)$. The final statement follows from an application of Proposition 4.55. $\square$

Lemma 4.57. *Let X be a connected CW-complex and X_k denote its k-skeleton. Suppose that X_0 is a single point and let $I_k = \ker(E^0(X) \xrightarrow{res} E^0(X_{k-1}))$. Then, for any i and j we have $I_i I_j \subseteq I_{i+j}$.*

Proof. Let $\Delta : X \to X \times X$ denote the diagonal map. Then, by standard topological arguments, Δ is homotopic to a skeleton-preserving map Δ', that is a map $\Delta' : X \to X \times X$ such that $\Delta'(X_k) \subseteq (X \times X)_k = \bigcup_{i+j=k} X_i \times X_j$. In fact, if i and j are any integers with $i + j = k$ we have $\Delta'(X_{k-1}) \subseteq (X_{i-1} \times X) \cup (X \times X_{j-1})$. Notice also that

$$(X \times X)/((X_{i-1} \times X) \cup (X \times X_{j-1})) = (X/X_{i-1}) \wedge (X/X_{j-1}).$$

Thus we get an induced map $\Delta' : X/X_{k-1} \to (X/X_{i-1}) \wedge (X/X_{j-1})$ fitting into the commutative diagram

$$\begin{tikzcd}
X_{k-1} \arrow[r, "\Delta'"] \arrow[d] & (X_{i-1} \times X) \cup (X \times X_{j-1}) \arrow[d] \\
X \arrow[r, "\Delta'"] \arrow[d] & X \times X \arrow[d] \\
X/X_{k-1} \arrow[r, "\Delta'"] & (X/X_{i-1}) \wedge (X/X_{j-1}).
\end{tikzcd}$$

On looking at the lower square and passing to cohomology we get

$$\begin{tikzcd}
E^0(X) & \\
\tilde{E}^0(X/X_{k-1}) \arrow[u] & \tilde{E}^0(X/X_{i-1}) \otimes_{E^0} \tilde{E}^0(X/X_{j-1}). \arrow[l] \arrow[ul]
\end{tikzcd}$$

Now, by consideration of the cofibre sequence $X_{k-1} \to X \to X/X_{k-1}$ and the associated long exact sequence in E-theory it follows that $I_k = \text{image}(\tilde{E}^0(X/X_{k-1}) \to E^0(X))$. Hence the above diagram gives the result. □

Lemma 4.58. *Let X be a connected CW-complex and X_k denote its k-skeleton. Suppose that X_0 is a single point and let $\mathfrak{m} = \ker(E^0(X) \xrightarrow{\epsilon} E^0 \xrightarrow{\pi} \mathbb{F}_p)$. Then $\mathfrak{m}$ is the unique maximal ideal of $E^0(X)$ and $E^0(X)$ is local.*

Proof. Since $E^0(X)/\mathfrak{m} \simeq \mathbb{F}_p$ we know that $\mathfrak{m}$ is maximal. Take $x \in E^0(X) \setminus \mathfrak{m}$. We will show that x is invertible. Note that x is non-zero mod $\mathfrak{m}$ and so $\epsilon(x)$ lies in $E^0 \setminus \mathfrak{m}_{E^0}$. Since E^0 is local it follows that $\epsilon(x)$ is invertible in E^0. Let $y = 1 - \epsilon(x)^{-1}x$. Then $y \in \ker(\epsilon) = I_1$ (where I_k $(k > 0)$ are the ideals of Lemma 4.57). It follows that $y^k \in I_k$ so that $z = \sum_{k=0}^{\infty} y^k$ converges in $E^0(X)$ with respect to the skeletal topology. But $1 + yz = z$ so that $(1 - y)z = 1$ and hence $1 - y = \epsilon(x)^{-1}x$ is invertible. It follows that x is invertible and $E^0(X)$ is local, as claimed. □

Lemma 4.59. *Let R be a Noetherian ring and A an algebra over R, finitely generated as an R-module. Then A is Noetherian.*

Proof. See, for example, [Sha00]. Every finitely generated R-module is Noetherian and every ideal of A is an R-submodule of A, so it follows that every ascending chain of ideals of A is necessarily eventually constant. □

Proposition 4.60. *Let G be a finite group. Then $E^0(BG)$ is a complete local Noetherian ring.*

Proof. By Lemma 4.59, $E^0(BG)$ is Noetherian as it is a finitely generated module over the Noetherian ring E^0. By Lemma 4.58 $E^0(BG)$ is local with maximal ideal $\mathfrak{m} = \ker(E^0(BG) \xrightarrow{\epsilon} E^0 \xrightarrow{\pi} \mathbb{F}_p)$. It remains to show that $E^0(BG)$ is complete with respect to the $\mathfrak{m}$-adic topology.

Now, $E^0(BG)$ inherits a topology from E^0, namely the $\mathfrak{m}_{E^0}$-adic topology generated by open balls of the form $a + \mathfrak{m}_{E^0}^r E^0(BG)$ and an application of the Artin-Rees lemma (specifically, Theorem 8.7 in [Mat89]) shows that $E^0(BG)$ is complete with respect to this topology. The discussion in [Mat89, p55] shows that the $\mathfrak{m}_{E^0}$-adic topology coincides with the $\mathfrak{m}$-adic one if and only if for each $N \in \mathbb{N}$ there exist r and s such that $\mathfrak{m}^r \subseteq \mathfrak{m}_{E^0}^N E^0(BG)$ and $\mathfrak{m}_{E^0}^s E^0(BG) \subseteq \mathfrak{m}^N$. Since $\mathfrak{m}_{E^0} E^0(BG) \subseteq \mathfrak{m}$ the latter of these conditions is easily satisfied and it remains to show that the former holds.

Let $I = I_1 = \ker(E^0(BG) \to E^0)$. Take the descending chain of ideals

$$E^0(BG) \rhd I \geqslant I^2 \geqslant I^3 \geqslant \dots$$

Then, writing $J_k = I^k/\mathfrak{m}_{E^0} I^k$, we get a descending chain of finite dimensional $\mathbb{F}_p$-vector spaces

$$J_1 \geqslant J_2 \geqslant J_3 \geqslant \dots$$

which must therefore be eventually constant. In particular, there exists $r \in \mathbb{N}$ with $J_{r+1} = J_r$ so that $I^{r+1}/\mathfrak{m}_{E^0} I^{r+1} = I^r/\mathfrak{m}_{E^0} I^r$. An application of Proposition 2.12 then gives $I^{r+1} = I^r$ whereby, since $I \subseteq \mathfrak{m}$, Nakayama's lemma leaves us with $I^r = 0$. But $E^0(BG)/I_1 = E^0$ and $E^0/\mathfrak{m}_{E^0} = \mathbb{F}_p$ so it follows that $\mathfrak{m} = I_1 + \mathfrak{m}_{E^0} E^0(BG)$ and $\mathfrak{m}^r \subseteq I_1^r + \mathfrak{m}_{E^0} E^0(BG) = \mathfrak{m}_{E^0} E^0(BG)$. Hence we see that $\mathfrak{m}^{rN} \subseteq \mathfrak{m}_{E^0}^N E^0(BG)$ and we are done. ☐

4.3.1 Transfers and the double coset formula

Given a finite group G and a subgroup H of G there is a map of E^*-modules $\mathrm{tr}_H^G : E^*(BH) \to E^*(BG)$ known as the *transfer map* generalising an analogous map for ordinary cohomology (see [Ben98]). The map is characterised by the following key properties.

Lemma 4.61. *Let G and G' be finite groups. Then the following hold.*

1. *If $K \leqslant H \leqslant G$ then $tr_K^G = tr_H^G \circ tr_K^H$.*
2. *If $H \leqslant G$ and $H' \leqslant G'$ then $tr_{H\times H'}^{G\times G'} = tr_H^G \otimes tr_{H'}^{G'}$ as maps $E^*(BH) \otimes E^*(BH') \to E^*(BG) \otimes E^*(BG')$.*
3. *(Frobenius Reciprocity) If $H \leqslant G$ and then $tr_H^G(res_H^G(a).b) = a.tr_H^G(b)$ for all $a \in E^*(BG)$, $b \in E^*(BH)$. That is, viewing $E^*(BH)$ as an $E^*(BG)$-module, tr_H^G is an $E^*(BG)$-module map.*
4. *If N is normal in G and π denotes the projection $G \to G/N$ then $tr_N^G(1) = \pi^* tr_1^{G/N}(1)$.*
5. *(Double Coset Formula) Suppose H and K are subgroups of G. Then, considered as maps $E^*(BK) \to E^*(BH)$, we have the identity*

$$res_H^G tr_K^G = \sum_{g \in H\backslash G/K} tr_{H\cap(gKg^{-1})}^H res_{H\cap(gKg^{-1})}^{gKg^{-1}} conj_g^*.$$

where $H \setminus G/K$ denotes the set of double cosets $\{HgK \mid g \in G\}$.

Proof. See [Ben91] and [Ben98]. ☐

There are many useful applications of this map, as we will see in subsequent sections.

Remark 4.62. Note that if H and K are subgroups of G with K normal in G then the double coset formula gives $\mathrm{res}^G_H \mathrm{tr}^G_K = \sum_{g \in H\backslash G/K} \mathrm{tr}^H_{H\cap K} \mathrm{res}^K_{H\cap K} \mathrm{conj}^*_g$. If, in addition, $H = K$ this simplifies further to leave us with $\mathrm{res}^G_H \mathrm{tr}^G_H = \sum_{g\in G/H} \mathrm{conj}^*_g$.

4.3.2 Further results in E-theory

Proposition 4.63. *Let $g \in G$ and write $conj_g$ for the conjugation map $G \to G$. Then $conj^*_g : E^*(BG) \to E^*(BG)$ is the identity map.*

Proof. This is a simple corollary to Proposition 2.7. □

Proposition 4.64. *Let G be a finite group and let H be a subgroup of G with $[G : H]$ coprime to p. Then the restriction map $E^*(BG) \to E^*(BH)$ is injective.*

Proof. Take $a \in E^*(BG)$. Then, by Frobenius reciprocity, we have $\mathrm{tr}^G_H(\mathrm{res}^G_H(a)) = a.\mathrm{tr}^G_H(1)$. Now, by an application of the double coset formula, $\mathrm{res}^G_1 \mathrm{tr}^G_H(1) = [G : H]$ which is coprime to p and hence a unit in E^0. Thus, reducing modulo the maximal ideal of $E^0(BG)$ we find that $\mathrm{tr}^G_H(1)$ maps to a unit in $\mathbb{F}_p^\times$ and so, by the general theory of local rings, it follows that it is a unit in $E^0(BG)$. Thus if $\mathrm{res}^G_H(a) = 0$ then $a = 0$, that is res^G_H is injective. □

Lemma 4.65. *Let H and K be subgroups of G with $K \subseteq N_G(H)$. Then K acts on $E^*(BH)$ and the map $E^*(BG) \to E^*(BH)$ lands in the K-invariants.*

Proof. Since $K \subseteq N_G(H)$, taking $k \in K$ we have a commutative diagram

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, "\mathrm{conj}_k"'] & G \arrow[d, "\mathrm{conj}_k"] \\ H \arrow[r, hook] & G \end{tikzcd}$$

and thus, since $\mathrm{conj}^*_k : E^*(BG) \to E^*(BG)$ is just the identity by Proposition 4.63, we get

$$\begin{tikzcd} E^*(BH) & E^*(BG) \arrow[l] \\ E^*(BH) \arrow[u, "\mathrm{conj}^*_k"] & E^*(BG) \arrow[l] \arrow[u, equal] \end{tikzcd}$$

showing that $E^*(BG) \to E^*(BH)$ lands in $E^*(BH)^K$. □

Proposition 4.66. *Let N be a normal subgroup of G with $[G : N]$ coprime to p. Then the restriction map induces an isomorphism $E^*(BG) \xrightarrow{\sim} E^*(BN)^{G/N}$.*

Proof. Since $[G : N]$ is coprime to p it follows from Proposition 4.64 that the restriction map $E^*(BG) \to E^*(BN)$ is injective. Further, by Lemma 4.65, the map lands in the G-invariants of $E^*(BN)$. But, since N acts trivially on $E^*(BN)$, the G-action on $E^*(BN)$ factors through a G/N action and thus we have an injective map $E^*(BG) \to E^*(BN)^{G/N}$.

For surjectivity, the double coset formula gives

$$\mathrm{res}^G_N \mathrm{tr}^G_N = \sum_{g \in N\backslash G/N} \mathrm{tr}^N_N \mathrm{res}^N_N \mathrm{conj}^*_g = \sum_{g\in G/N} \mathrm{conj}^*_g$$

(where we have used the fact that N is normal) so that if $a \in E^*(BN)^{G/N}$ we get

$$\mathrm{res}_N^G \mathrm{tr}_N^G(a) = \sum_{g \in G/N} \mathrm{conj}_g^*(a) = |G/N|a.$$

Thus, since $|G/N|$ is coprime to p and hence a unit in E^*, we have

$$\mathrm{res}_N^G \left(\frac{1}{|G/N|} \mathrm{tr}_N^G(a) \right) = \frac{1}{|G/N|}.|G/N|a = a$$

so that $\mathrm{res}_N^G : E^*(BG) \to E^*(BN)^{G/N}$ is surjective, as required. □

4.3.3 Understanding $E^*(BG)$ as a categorical limit

For a finite group G let $\mathcal{A}(G)$ be the category with objects the abelian subgroups of G and morphisms from B to A being the maps of G-sets $G/B \to G/A$. These can be understood as shown below.

Proposition 4.67. *Let G be a finite group and A and B be abelian subgroups of G. If $f : G/B \to G/A$ is a map of G-sets then f is determined by $f(B)$ and, writing $f(B) = gA$, we have $g^{-1}Bg \subseteq A$.*

Conversely, if there is $g \in G$ with $g^{-1}Bg \subseteq A$ then there is a map of G-sets $G/B \to G/A$ given by $f(B) = gA$.

Proof. Let $f : G/B \to G/A$ be a map of G-sets. Then, for all $g \in G$, we have $f(gB) = gf(B)$ so that f is determined by $f(B)$. Write $f(B) = gA$. Then for all $b \in B$ we have

$$gA = f(B) = f(bB) = bf(B) = bgA$$

so that $g^{-1}bg \in A$. Hence $g^{-1}Bg \subseteq A$.

For the converse, if $g \in G$ with $g^{-1}Bg \subseteq A$ the G-map $f : G/B \to G/A$ given by $f(B) = gA$ is well defined since if $hB = kB$ we have $k = hb$ for some $b \in B$ whereby

$$f(kB) = kf(B) = hbf(B) = hbgA = hg(g^{-1}bg)A = hgA = hf(B) = f(hB).$$ □

The following result is due to Hopkins, Kuhn and Ravenel ([HKR00, Theorem A]).

Theorem 4.68. *Let E be a complex oriented cohomology theory and G be a finite group. Then the map*

$$\frac{1}{|G|}E^*(BG) \to \lim_{A \in \mathcal{A}(G)} \frac{1}{|G|}E^*(BA)$$

is an isomorphism.

Remark 4.69. Note that $\lim_{A \in \mathcal{A}(G)} \frac{1}{|G|}E^*(BA)$ is a subring of $\prod_{A \in \mathcal{A}(G)} \frac{1}{|G|}E^*(BA)$. Note also that E^* is p-local and $E^*(BG)$ is trivial if $p \nmid |G|$. It follows that inverting the order of G can be replaced by inverting p or, equivalently, by tensoring with $\mathbb{Q}$.

In fact we can simplify the right-hand limit somewhat. We will use the following modified version of the theorem.

Proposition 4.70. *Let $\mathcal{A}(G)_{(p)}$ denote the full subcategory of $\mathcal{A}(G)$ consisting of the abelian p-subgroups of G. Then the map*

$$\mathbb{Q} \otimes E^*(BG) \to \lim_{A \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^*(BA)$$

is an isomorphism.

Proof. This was remarked on in [HKR00, Remark 3.5]. By abstract category theory there is a unique map $\lim_{A \in \mathcal{A}(G)} \mathbb{Q} \otimes E^*(BA) \to \lim_{A \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^*(BA)$ commuting with the arrows. We will show this is an isomorphism.

Given any $C \in \mathcal{A}(G)$ we have $C_{(p)} \in \mathcal{A}(G)_{(p)}$ and hence, using the fact that $\mathrm{res}^C_{C_{(p)}}$ is an isomorphism (Proposition 4.52), a composite map

$$\lim_{A \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^*(BA) \to \mathbb{Q} \otimes E^*(BC_{(p)}) \xrightarrow{\sim} \mathbb{Q} \otimes E^*(BC).$$

Now, take any map $C \to D$ in $\mathcal{A}(G)$ corresponding to an element $g \in G$. Then g induces a morphism $C_{(p)} \to D_{(p)}$ and we have the following commutative diagram.

$$\begin{tikzcd}
 & \mathbb{Q} \otimes E^*(BD_{(p)}) \arrow[r, "\sim"] \arrow[dd, "\mathrm{conj}_g^*"'] & \mathbb{Q} \otimes E^*(BD) \arrow[dd, "\mathrm{conj}_g^*"] \\
\lim_{A \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^*(BA) \arrow[ur] \arrow[dr] \arrow[urr, dashed, bend left] \arrow[drr, dashed, bend right] & & \\
 & \mathbb{Q} \otimes E^*(BC_{(p)}) \arrow[r, "\sim"'] & \mathbb{Q} \otimes E^*(BC).
\end{tikzcd}$$

Thus by category theory we have a map $\lim_{A \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^*(BA) \to \lim_{A \in \mathcal{A}(G)} \mathbb{Q} \otimes E^*(BA)$ which must be the inverse to our original map. □

Before a corollary, we make the following definition.

Definition 4.71. Let $f_i : R \to S_i$ $(i \in I)$ be a family of maps. Then we say that the maps $(f_i)_{i \in I}$ are *jointly injective* if the map $\prod_i f_i : R \to \prod_i S_i$ is injective.

Corollary 4.72. *Let G be a finite group with $E^*(BG)$ free over E^*. Let $A_1, \dots, A_s$ be abelian subgroups of G such that for each abelian p-subgroup A of G there is $g \in G$ such that $gAg^{-1} \subseteq A_i$ for some i. Then the restriction maps $E^*(BG) \to E^*(BA_i)$ are jointly injective.*

Proof. Take $a \in E^*(BG)$ and suppose that a maps to 0 in $\prod_{i=1}^s E^*(BA_i)$, that is a maps to 0 in each $E^*(BA_i)$. Take any abelian p-subgroup A of G. Then, by the hypothesis, we have $g \in G$ with $gAg^{-1} \subseteq A_i$ for some i. Hence we get

$$\begin{tikzcd}
E^*(BG) \arrow[r, "\mathrm{res}"] \arrow[d, equal, "\mathrm{conj}_g^*"'] & E^*(BA_i) \arrow[r, "\mathrm{res}"] & E^*(B(gAg^{-1})) \arrow[d, "\mathrm{conj}_g^*", "\wr"'] \\
E^*(BG) \arrow[rr, "\mathrm{res}"] & & E^*(BA)
\end{tikzcd}$$

showing that a maps to 0 in $E^*(BA)$. Thus a maps to 0 in $\lim_{A \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^*(BA)$ and hence in $\mathbb{Q} \otimes E^*(BG)$ by Proposition 4.70. But $E^*(BG)$ is free over E^* so that $E^*(BG) \to \mathbb{Q} \otimes E^*(BG)$ is injective. Thus $a = 0$, as required. □

4.3.4 Hopkins, Kuhn and Ravenel's good groups.

We use the following concept from [HKR00].

Definition 4.73. Let G be a finite group. Then we define G to be *good* if $K(n)^*(BG)$ is generated over $K(n)^*$ by transfers of Euler classes of complex representations of subgroups of G. Note that if G is good then $K(n)^*(BG)$ is concentrated in even degrees and Proposition 4.56 applies.

Hence for good groups the only interesting cohomology is $E^0(BG)$ and we can recover $K^0(BG)$ as $K^0 \otimes_{E^0} E^0(BG)$. The following is Theorem E from [HKR00].

Proposition 4.74. *Using the terminology of Definition 4.73,*

1. *every finite abelian group is good;*
2. *if G_1 and G_2 are good then so is $G_1 \times G_2$;*
3. *if $Syl_p(G)$ is good then so is G;*
4. *if G is good then so is the wreath product $C_p \wr G$.*

We apply the results to our finite general linear groups to get the following.

Proposition 4.75. *Let K be a finite field with $v_p(|K^\times|) > 0$. Then, for any $d > 0$, the finite group $GL_d(K)$ is good.*

Proof. By Proposition 4.74 part 3 it is sufficient to show that the Sylow p-subgroup of $GL_d(K)$ is good. By Proposition 3.12 we know that $\mathrm{Syl}_p(GL_d(K)) = P_0 \wr P_1$ where $P_0 = \mathrm{Syl}_p(\Sigma_d)$ and P_1 is the p-part of $K^\times$. Clearly P_1 is good by Proposition 4.74 part 1. Hence if $d < p$ then $\mathrm{Syl}_p(GL_d(K)) = P_1$, which is good.

If $d = p^k$ for some $k > 0$ then, by Proposition 3.3, P_0 is an iterated wreath product of copies of C_p so that $P_0 \wr P_1 = C_p \wr \ldots \wr C_p \wr P_0$ which is good by repeated use of Proposition 4.74 part 4.

For arbitrary d we write $d = \sum_i a_i p^i$ and then, by Proposition 3.4, $P_0 = \prod_i \mathrm{Syl}_p(\Sigma_{p^i})^{a_i}$ with each $\mathrm{Syl}_p(\Sigma_{p^i})^{a_i}$ an iterated wreath product of copies of C_p. Thus, using Lemma 2.6, we have

$$P_0 \wr P_1 = \left(\prod_i \mathrm{Syl}_p(\Sigma_{p^i})^{a_i} \right) \wr P_1 = \prod_i \left(\mathrm{Syl}_p(\Sigma_{p^i}) \wr P_1 \right)^{a_i}$$

with each term in the product good. Thus $P_0 \wr P_1$ is good. □

4.3.5 The cohomology of general linear groups over algebraically closed fields

Our main starting point for the cohomology of the finite general linear groups comes from the well understood theory of general linear groups over the relevant algebraically closed fields. For any prime l different to p we briefly outline the relevant results.

Proposition 4.76. *Let $T \simeq S^1 \times \ldots \times S^1$ denote the maximal torus of $GL_d(\mathbb{C})$. Then restriction induces an isomorphism $H^*(BGL_d(\mathbb{C}); \mathbb{Z}_{(p)}) \xrightarrow{\sim} H^*(BT; \mathbb{Z}_{(p)})^{\Sigma_d}$.*

Proof. This is a classical theorem of Borel ([Bor53]). $\square$

Corollary 4.77. *With the above notation, the restriction map gives an isomorphism*

$$E^*(BGL_d(\mathbb{C})) \xrightarrow{\sim} E^*(BT)^{\Sigma_d}.$$

Proof. The proof is analogous to that of [Tan95, Corollary 2.10]. $\square$

Proposition 4.78. *Let h be any multiplicative cohomology theory for which $p = 0$ in h^* and let l be a prime different to p. Then, writing $\overline{T}_d$ for the maximal torus of $GL_d(\overline{\mathbb{F}}_l)$, there are compatible maps $B\overline{T}_d \to BT$ and $BGL_d(\overline{\mathbb{F}}_l) \to BGL_d(\mathbb{C})$ which induce isomorphisms $h^*(BT) \to h^*(B\overline{T}_d)$ and $h^*(BGL_d(\mathbb{C})) \xrightarrow{\sim} h^*(BGL_d(\overline{\mathbb{F}}_l))$.*

Proof. The main result of [FM84] gives maps which induce an isomorphism on mod-p homology and hence also on mod-p cohomology. The proof then relies on a comparison of the relevant Atiyah-Hirzebruch spectral sequences $H^*(X; h^*) \implies h^*(X)$. $\square$

Lemma 4.79. *Let X be a CW-complex with X_0 a single point. Then if $K(n)^*(X) = 0$ we have $E^*(X) = 0$.*

Proof. If $K(n)^*(X) = 0$ then $K(n)^*(X)$ is (trivially) finitely generated over $K(n)^*$. Proposition 2.4 of [HS99] then tells us that $E^*(X)$ is finitely generated over E^*. Further, $K(n)^*(X)$ is (trivially) concentrated in even degrees so Proposition 2.5 of [HS99] tells us that $E^*(X)$ is concentrated in even degrees and $0 = K(n)^0(X) = E^0(X)/\mathfrak{m}_{E^0}E^0(X)$. But $E^0(X)$ is local by Proposition 4.58 and an application of Nakayama's lemma gives $E^0(X) = 0$. It follows that $E^*(X) = 0$. $\square$

Corollary 4.80. *Let X and Y be spaces as in Lemma 4.79. Let $f : X \to Y$ be such that the induced map $f^* : K(n)^*(Y) \to K(n)^*(X)$ is an isomorphism. Then $f^* : E^*(Y) \to E^*(X)$ is also an isomorphism.*

Proof. The cofibre sequence $X \to Y \to Y/X$ gives, on passing to cohomology, $K(n)^*(X/Y) = 0$ since $K(n)^*(Y) \to K(n)^*(X)$ is an isomorphism. Hence, by Lemma 4.79, we find that $E^*(X/Y) = 0$ and, reversing the previous argument, an isomorphism $E^*(Y) \to E^*(X)$. $\square$

Proposition 4.81. *The maps of Proposition 4.78 give rise to compatible isomorphisms $E^*(BT) \simeq E^*(B\overline{T}_d)$ and $E^*(BGL_d(\mathbb{C})) \simeq E^*(BGL_d(\overline{\mathbb{F}}_l))$. Hence restriction induces an isomorphism $E^*(BGL_d(\overline{\mathbb{F}}_l)) \simeq E^*(B\overline{T}_d)^{\Sigma_d}$.*

Proof. By Proposition 4.78 we know that we get such maps in $K(n)$-theory. But an application of Corollary 4.80 shows that we get the same result in E-theory. $\square$

Now, recall from Section 2.1.4 that we have a chosen embedding $\overline{\mathbb{F}}_l^\times \to S^1$. Let $x \in E^2(B\overline{\mathbb{F}}_l^\times)$ be the restriction of the complex orientation for E under the induced map $E^*(\mathbb{C}P^\infty) \to E^*(B\overline{\mathbb{F}}_l^\times)$. Write π_i for the i^{th} projection $\overline{T}_d \simeq (\overline{\mathbb{F}}_l^\times)^d \to \overline{\mathbb{F}}_l^\times$. Proposition 4.81 then gives us following corollary.

Corollary 4.82. *Restriction induces an isomorphism*

$$E^*(BGL_d(\overline{\mathbb{F}}_l)) \xrightarrow{\sim} E^*(B\overline{T}_d)^{\Sigma_d} = E^*[\![x_1, \ldots, x_d]\!]^{\Sigma_d} = E^*[\![\sigma_1, \ldots, \sigma_d]\!]$$

where σ_i is the i^{th} elementary symmetric function in $x_1 = \pi_1^*(x), \ldots, x_d = \pi_d^*(x)$.

4.3.6 The cohomology of the symmetric group Σ_p

Using the analysis of Section 3.1.1 we can get an understanding of the cohomology of Σ_p.

Proposition 4.83. *Let $C_p = \langle \gamma_p \rangle$ denote standard cyclic subgroup of Σ_p of order p. Then the inclusion $C_p \rightarrowtail \Sigma_p$ induces an isomorphism $E^0(B\Sigma_p) \simeq E^0(BC_p)^{Aut(C_p)}$.*

Proof. Let $M = \mathrm{Aut}(C_p) \ltimes C_p \leqslant \Sigma_p$. Then a transfer argument gives an isomorphism $H^*(B\Sigma_p; \mathbb{F}_p) \xrightarrow{\sim} H^*(BM; \mathbb{F}_p)$ (see, for example, [Ben98, p74]). Hence, an application of the Atiyah-Hirzebruch spectral sequence shows that restriction gives a $K(n)^*$-isomorphism and, by Corollary 4.80, we get an isomorphism $E^0(B\Sigma_p) \xrightarrow{\sim} E^0(BM)$. But C_p is normal in M and $|M/C_p| = |\mathrm{Aut}(C_p)| = p-1$ which is coprime to p so that we can use Proposition 4.66 to get $E^0(BM) \xrightarrow{\sim} E^0(BC_p)^{\mathrm{Aut}(C_p)}$. Hence $E^0(B\Sigma_p) \simeq E^0(BC_p)^{\mathrm{Aut}(C_p)}$. □

Lemma 4.84. *In $E^0[\![x]\!]/[p](x)$ we have $[k](x) = [\hat{k}](x)$ for all $k \in (\mathbb{Z}/p)^\times$.*

Proof. We know that $\hat{k} = k \bmod p$, so that $k = \hat{k} + ap$ for some $a \in \mathbb{Z}_p$. Then, modulo $[p](x)$, we have $[k](x) = [\hat{k} + ap](x) = [\hat{k}](x) +_F [ap](x) = [\hat{k}](x) +_F [a]([p](x)) = [\hat{k}](x)$, as claimed. □

Lemma 4.85. *With the obvious embedding $C_p \rightarrowtail S^1$, let x denote the corresponding generator of $E^0(BC_p)$. Then $x^{p-1} \in E^0(BC_p)^{Aut(C_p)}$.*

Proof. First note that $\mathrm{Aut}(C_p) \simeq (\mathbb{Z}/p)^\times$ acts on $E^0(BC_p)$ by $k.x = [k](x)$. It follows that $\prod_{k=1}^{p-1}[k](x) \in E^0(BC_p)^{\mathrm{Aut}(C_p)}$. But, by Lemmas 4.84 and 2.3, we have $\prod_{k=1}^{p-1}[k](x) = \prod_{k=1}^{p-1}[\hat{k}](x) = \prod_{k=1}^{p-1}\hat{k}x = -x^{p-1}$ so that $x^{p-1} = -\prod_{k=1}^{p-1}[k](x) \in E^0(BC_p)^{\mathrm{Aut}(C_p)}$. □

Proposition 4.86. *Put $d = -x^{p-1} \in E^0(BC_p)^{Aut(C_p)}$. Then $E^0(BC_p)^{Aut(C_p)}$ is free over E^0 with basis $\{1, d, \ldots, d^{(p^n-1)/(p-1)}\}$.*

Proof. We have a basis $\{1, x, \ldots, x^{p^n-1}\}$ for $E^0(BC_p)$ over E^0. Thus, for $k \in \mathrm{Aut}(C_p) \simeq (\mathbb{Z}/p)^\times$, we have $k.x^i = [k](x)^i = [\hat{k}](x)^i = \hat{k}^i x^i$. Taking any $\sum_i a_i x^i \in E^0(BC_p)$, for all $k \in \mathrm{Aut}(C_p)$ we have

$$\begin{aligned}
\textstyle\sum_i a_i x^i \in E^0(BC_p)^{\mathrm{Aut}(C_p)} \quad &\Longleftrightarrow \quad \textstyle\sum_i a_i x^i = k.\sum_i a_i x^i \\
&\Longleftrightarrow \quad \textstyle\sum_i a_i x^i = \sum_i a_i \hat{k}^i x^i \\
&\Longleftrightarrow \quad a_i \hat{k}^i = a_i \quad \text{for all } i \\
&\Longleftrightarrow \quad a_i(\hat{k}^i - 1) = 0 \quad \text{for all } i.
\end{aligned}$$

But $\hat{k}^i = 1$ if and only if $i = 0 \bmod p-1$. Hence $\sum_i a_i x^i \in E^0(BC_p)^{\mathrm{Aut}(C_p)}$ if and only if $a_i = 0$ for $i \neq 0 \bmod p-1$. Thus

$$E^0(BC_p)^{\mathrm{Aut}(C_p)} = E^0\{x^{j(p-1)} \mid 0 \le j \le \tfrac{p^n-1}{p-1}\} = E^0\{1, d, \ldots, d^{(p^n-1)/(p-1)}\}.$$ □

Lemma 4.87. *With x and d as above, $\langle p \rangle(x) \in E^0(BC_p)^{Aut(C_p)}$. Hence there is a polynomial $f(t) \in E^0[t]$ such that $\langle p \rangle(x) = f(d)$. Further, $f(0) = p$.*

Proof. Taking $k \in \mathrm{Aut}(C_p) \simeq (\mathbb{Z}/p)^\times$ we have $k.\langle p\rangle(x) = \langle p\rangle([k](x)) = \langle p\rangle([\hat{k}](x)) = \langle p\rangle(x)$ by Lemma 4.34. Thus, using Proposition 4.86 we can write $\langle p\rangle(x) = \sum_i a_i d^i = f(d)$ for some $a_i \in E^0$, as claimed. Putting $x = 0$ then gives $f(0) = \langle p\rangle(0) = p$. $\square$

Proposition 4.88. *Let x, d and f be as above. Then $E^0(BC_p)^{Aut(C_p)} \simeq E^0[\![d]\!]/df(d)$.*

Proof. Firstly, note that we have $df(d) = -x^{p-1}\langle p\rangle(x) = 0$ in $E^0(BC_p) = E^0[\![x]\!]/x\langle p\rangle(x)$. Take any $g(t) \in E^0[\![t]\!]$ with $g(d) = 0$ in $E^0(BC_p)$. Then $g(d) = h(x)[p](x)$ in $E^0[\![x]\!]$ for some h. Putting $x = 0$ we see that $g(0) = h(0)[p](0) = 0$ and so $d|g(d) = h(x)[p](x) = xh(x)\langle p\rangle(x) = xh(x)f(d)$. Since $f(0) = p \neq 0$ we conclude that $d|xh(x)$. Hence $g(d) \in df(d)E^0(BC_p)$; that is, $g(d) = df(d)k_0(x)$ for some $k_0(x)$. But then $k(x) = \frac{1}{p-1}\sum_{\alpha\in\mathrm{Aut}(C_p)} \alpha.k_0(x)$ is $\mathrm{Aut}(C_p)$-invariant and $g(d) = df(d)k(x) \in df(d)E^0(BC_p)^{\mathrm{Aut}(C_p)}$, as required. $\square$

We will need the following standard results later.

Lemma 4.89. *Let x, d and f be as above. Then $tr_1^{C_p}(1) = \langle p\rangle(x)$ and $tr_1^{\Sigma_p}(1) = (p-1)!f(d)$.*

Proof. Write $\mathrm{tr}_1^{C_p}(1) = g(x) \bmod [p](x)$ for some $g(x) \in E^0[\![x]\!]$. Then, using Frobenius reciprocity (Lemma 4.61) we have $x.\mathrm{tr}_1^{C_p}(1) = \mathrm{tr}_1^{C_p}(\mathrm{res}_1^{C_p}(x).1) = 0$ so that we must have $xg(x) = 0 \bmod [p](x)$. Thus $xg(x) = x\langle p\rangle(x)h(x)$ for some h and hence $g(x) = \langle p\rangle(x)h(x)$. Thus, mod $[p](x)$, we find that $g(x) = \langle p\rangle(x)h(0)$. But $g(0) = \mathrm{res}_1^{C_p}\mathrm{tr}_1^{C_p}(1) = |C_p| = p$ by an application of the double-coset formula. Hence $p = g(0) = \langle p\rangle(0)h(0) = ph(0)$ so that $h(0) = 1$. Thus $\mathrm{tr}_1^{C_p}(1) = \langle p\rangle(x)$, as claimed. For the second statement, we have

$$\begin{aligned}
\mathrm{res}_{C_p}^{\Sigma_p}\mathrm{tr}_1^{\Sigma_p}(1) &= \sum_{\sigma\in C_p\backslash\Sigma_p/1} \mathrm{tr}_{C_p\cap 1}^{C_p}\mathrm{res}_{C_p\cap 1}^{1}(\mathrm{conj}_\sigma^*(1)) \\
&= \sum_{\sigma\in C_p\backslash\Sigma_p} \mathrm{tr}_1^{C_p}(1) \\
&= (p-1)!\langle p\rangle(x).
\end{aligned}$$

But $\mathrm{res}_{C_p}^{\Sigma_p}$ is injective so we find that $\mathrm{tr}_1^{\Sigma_p}(1) = (p-1)!\langle p\rangle(x) = (p-1)!f(d)$. $\square$

4.3.7 l-Chern classes

The results of Section 4.3.5 allow us to construct convenient cohomology classes which are analogous to the construction of ordinary Chern classes.

Definition 4.90. Let G be a group. Recall that $E^0(BGL_d(\overline{\mathbb{F}}_l)) \simeq E^0[\![\sigma_1, \ldots, \sigma_d]\!]$. Given a group homomorphism $\alpha : G \to GL_d(\overline{\mathbb{F}}_l)$ (equivalently, a finite-dimensional $\overline{\mathbb{F}}_l$-representation of G) and a natural number k we define the k^{th} *l-Chern class of α* by

$$\hat{c}_k(\alpha) = \begin{cases} \alpha^*(\sigma_k) & \text{for } 0 \le k \le d \\ 0 & \text{otherwise,} \end{cases}$$

(where σ_0 is understood to be 1).

Proposition 4.91. *The l-Chern classes satisfy the following properties.*

1. *For any group homomorphism $\alpha : G \to GL_d(\overline{\mathbb{F}}_l)$ we have $\hat{c}_0(\alpha) = 1$.*

2. *(Functoriality) Given group homomorphisms $\alpha : G \to GL_d(\overline{\mathbb{F}}_l)$ and $f : H \to G$ we get $\hat{c}_k(\alpha \circ f) = f^*(\hat{c}_k(\alpha))$ for all k.*

3. *Given group homomorphisms* $\alpha : G \to GL_{d_1}(\overline{\mathbb{F}}_l)$ *and* $\beta : G \to GL_{d_2}(\overline{\mathbb{F}}_l)$ *we get*

$$\hat{c}_k(\alpha \oplus \beta) = \sum_{i+j=k} \hat{c}_i(\alpha)\hat{c}_j(\beta)$$

where $\alpha \oplus \beta : G \to GL_{d_1+d_2}(\overline{\mathbb{F}}_l)$ *is given by* $g \mapsto \alpha(g) \oplus \beta(g)$.

4. *Let* $id_l : \overline{\mathbb{F}}_l^{\times} \to \overline{\mathbb{F}}_l^{\times}$ *be the identity. Then* $\hat{c}_1(id_l) = x$*, the restriction of our complex orientation.*

Proof. Properties 1 and 4 follow straight from the definition. For property 2, if $k \leq 0$ or $k > d$ the result is clear since $f^*(0) = 0$ and $f^*(1) = 1$. Otherwise $1 \leq k \leq d$ and we have

$$\hat{c}_k(\alpha \circ f) = (\alpha \circ f)^*(\sigma_k) = (f^* \circ \alpha^*)(\sigma_k) = f^*(\alpha^*(\sigma_k)) = f^*(\hat{c}_k(\alpha)),$$

as required. It remains to prove property 3.

We note first that the diagram

$$\begin{tikzcd}
(\overline{\mathbb{F}}_l^{\times})^{d_1} \times (\overline{\mathbb{F}}_l^{\times})^{d_2} \arrow[r, "\sim"] \arrow[d] & (\overline{\mathbb{F}}_l^{\times})^{d_1+d_2} \arrow[d] \\
GL_{d_1}(\overline{\mathbb{F}}_l) \times GL_{d_2}(\overline{\mathbb{F}}_l) \arrow[r, "\mu"] & GL_{d_1+d_2}(\overline{\mathbb{F}}_l)
\end{tikzcd}$$

induces

$$\begin{tikzcd}
E^0(B(\overline{\mathbb{F}}_l^{\times})^{d_1})\widehat{\otimes}_{E^0} E^0(B(\overline{\mathbb{F}}_l^{\times})^{d_2}) & E^0(B(\overline{\mathbb{F}}_l^{\times})^{d_1+d_2}) \arrow[l, "\sim"'] \\
E^0(BGL_{d_1}(\overline{\mathbb{F}}_l))\widehat{\otimes}_{E^0} E^0(BGL_{d_2}(\overline{\mathbb{F}}_l)) \arrow[u] & E^0(BGL_{d_1+d_2}(\overline{\mathbb{F}}_l)). \arrow[l, "\mu^*"'] \arrow[u]
\end{tikzcd}$$

In the usual way, we can write $E^0(B(\overline{\mathbb{F}}_l^{\times})^{d_1+d_2}) \simeq E^0[\![x_1, \ldots, x_{d_1}, x_{d_1+1}, \ldots, x_{d_1+d_2}]\!]$ to get $E^0(B(\overline{\mathbb{F}}_l^{\times})^{d_1}) \simeq E^0[\![x_1, \ldots, x_{d_1}]\!]$ and $E^0(B(\overline{\mathbb{F}}_l^{\times})^{d_2}) \simeq E^0[\![x_{d_1+1}, \ldots, x_{d_1+d_2}]\!]$ where $x_i = \hat{c}_1((\overline{\mathbb{F}}_l^{\times})^{d_1+d_2} \xrightarrow{\pi_i} \overline{\mathbb{F}}_l^{\times})$. We use the following lemma.

Lemma 4.92. *Let* $d_1, d_2 \in \mathbb{N}$ *and* $d = d_1 + d_2$*. For* $1 \leq k \leq d$ *let* σ_k *be the* k^{th} *elementary symmetric function in* $x_1, \ldots, x_d$*. Let* $\sigma_{d_1,i}$ *(respectively* $\sigma_{d_2,j}$*) be the* i^{th} *(respectively* j^{th}*) elementary symmetric function in* $x_1, \ldots, x_d$ *(respectively* $x_{d_1+1}, \ldots, x_d$*). Then*

$$\sigma_k = \sum_{i+j=k} \sigma_{d_1,i}\sigma_{d_2,j}.$$

Proof. Straightforward combinatorics. ∎

Using the notation of Lemma 4.92 we then have $E^0(BGL_{d_1}(\overline{\mathbb{F}}_l)) = E^0[\![\sigma_{d_1,1} \ldots, \sigma_{d_1,d_1}]\!]$ and $E^0(BGL_{d_2}(\overline{\mathbb{F}}_l)) = E^0[\![\sigma_{d_2,1} \ldots, \sigma_{d_2,d_2}]\!]$. Thus, chasing the diagram and using injectivity of the vertical arrows, we see that $\mu^*(\sigma_k) = \sum_{i+j=k} \sigma_{d_1,i} \otimes \sigma_{d_2,j}$.

Now given homomorphisms $\alpha : G \to GL_{d_1}(\overline{\mathbb{F}}_l)$ and $\beta : G \to GL_{d_2}(\overline{\mathbb{F}}_l)$, the map $\alpha \oplus \beta$ is given by the composition $G \xrightarrow{\alpha\times\beta} GL_{d_1}(\overline{\mathbb{F}}_l) \times GL_{d_2}(\overline{\mathbb{F}}_l) \xrightarrow{\mu} GL_d(\overline{\mathbb{F}}_l)$, whereby

$$\begin{aligned}
\hat{c}_k(\alpha \oplus \beta) &= (\alpha \oplus \beta)^* \mu^*(\sigma_k) \\
&= (\alpha \oplus \beta)^* \left(\sum_{i+j=k} \sigma_{d_1,i} \otimes \sigma_{d_2,j} \right) \\
&= \sum_{i+j=k} \alpha^*(\sigma_{d_1,i}) \otimes \beta^*(\sigma_{d_2,j}) \\
&= \sum_{i+j=k} \hat{c}_i(\alpha)\hat{c}_j(\beta).
\end{aligned}$$

□

Remark 4.93. The properties of Proposition 4.91 are enough to completely determine the l-Chern classes. That is, they can be viewed as axioms for the l-Chern classes.

Proposition 4.94. *Given one-dimensional $\overline{\mathbb{F}}_l$-representations $\alpha, \beta : G \to \overline{\mathbb{F}}_l^\times$ we have*

$$\hat{c}_1(\alpha \otimes \beta) = \hat{c}_1(\alpha) +_F \hat{c}_1(\beta).$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "{(\alpha,\beta)}"] \arrow[dr, "\alpha\otimes\beta"'] & \overline{\mathbb{F}}_l^\times \times \overline{\mathbb{F}}_l^\times \arrow[d, "\mu"] \\
& \overline{\mathbb{F}}_l^\times
\end{tikzcd}$$

which, on passing to cohomology, gives the result. □

Definition 4.95. Given a group homomorphism $\alpha : G \to GL_d(\overline{\mathbb{F}}_l)$ we define the *l-Euler class of* α to be the top l-Chern class, that is $\mathrm{euler}_l(\alpha) = \hat{c}_d(\alpha)$.

Chapter 5

Generalised character theory

5.1 Generalised characters

In this section we outline the generalised character theory developed by Hopkins, Kuhn and Ravenel in [HKR00]. We start by recalling some results on Pontryagin duality.

5.1.1 Locally compact groups

A topological group G is called *locally compact* if there is a neighbourhood of the identity which is contained in a compact set. Given a locally compact abelian group G we define its *Pontryagin dual* or *character group*, G^*, by $G^* = \mathrm{Hom}_{\mathrm{cts}}(G, S^1)$. We give G^* the weakest topology such that the maps $G^* \to S^1, \chi \mapsto \chi(g)$ are continuous for each $g \in G$. If G is locally compact then G^* is also locally compact and the assignment $G \mapsto G^*$ is a contravariant endofunctor on locally compact abelian groups. Further, for each G there is a canonical (continuous) isomorphism $G \to (G^*)^*$ given by $g \mapsto (G^* \to S^1, \chi \mapsto \chi(g))$. From here on G may be identified with $(G^*)^*$ without comment.

Every discrete (and hence every finite) group is locally compact. Other examples of locally compact groups include $\mathbb{R}, S^1$ and $\mathbb{Z}_p$. If G and H are locally compact then $G \oplus H$ is locally compact. Using the result $\mathrm{Hom}(\varinjlim A_i, B) = \varprojlim \mathrm{Hom}(A_i, B)$ from general category theory, we see that $(\varinjlim A_i)^* = \varprojlim (A_i)^*$ wherever the limits exist. Of particular interest to us will be the result $(\mathbb{Z}/p^\infty)^* = (\varinjlim \mathbb{Z}/p^r)^* = \varprojlim (\mathbb{Z}/p^r)^* = \mathbb{Z}_p$.

For each $m > 1$ there is a canonical isomorphism $\mathbb{Z}/m \xrightarrow{\sim} (\mathbb{Z}/m)^*$ given by $1 \mapsto (\mathbb{Z}/m \to S^1, 1 \mapsto e^{2\pi i/m})$. Further, if G and H are locally compact abelian groups then

$$(G \oplus H)^* = \mathrm{Hom}_{\mathrm{cts}}(G \oplus H, S^1) \simeq \mathrm{Hom}_{\mathrm{cts}}(G, S^1) \oplus \mathrm{Hom}_{\mathrm{cts}}(H, S^1) = G^* \oplus H^*.$$

Hence, if G is any finite abelian group then there is an isomorphism $G \simeq G^*$ but in general this will be non-canonical.

We will mainly be looking at a group $\Theta \simeq (\mathbb{Z}/p^\infty)^n$. From the remarks earlier it is clear that $\Theta^* \simeq \mathbb{Z}_p^n$.

5.1.2 Generalised characters

As usual, let $x \in E^0(\mathbb{C}P^\infty)$ be our standard complex coordinate and F the associated standard p-typical formal group law. Recall that for each $m \geq 0$ we let $g_m(t)$ denote the Weierstrass polynomial of degree p^{nm} which is a unit multiple of $[p^m]_F(t)$ in $E^0[\![t]\!]$.

Let $Q(E^0)$ denote the field of fractions of E^0 and fix an algebraic closure $\overline{Q(E^0)}$. For each $m \geq 0$ let $\Theta_m = \{a \in \overline{Q(E^0)} \mid g_m(a) = 0\}$. Note that $\Theta_m \subseteq \Theta_{m+1}$ and define $\Theta = \bigcup_m \Theta_m$.

Proposition 5.1. *There are abelian group structures on Θ_m and Θ given by $+_F$ and (non-canonical) isomorphisms $\Theta_m \simeq (\mathbb{Z}/p^m)^n$ and $\Theta \simeq (\mathbb{Z}/p^\infty)^n$.*

Proof. If $a, b \in \Theta_m$ then $[p^m](a +_F b) = [p^m](a) +_F [p^m](b) = 0$, so Θ_m is closed under $+_F$. The identity element is 0. Given $a \in \Theta_m$ we have $[p^m](-_F a) = -_F[p^m](a) = 0$ which gives inverses. Finally, associativity and commutativity follow easily from properties of $+_F$. Thus Θ_m is an abelian group for all m and hence so is $\Theta = \bigcup_m \Theta_m$.

The roots of g_m are distinct, so that $|\Theta_m| = p^{mn}$ (see, for example, [Str97, Proposition 27]). Hence Θ_m is a finite abelian p-group and we can write $\Theta_m \simeq \mathbb{Z}/p^{r_1} \oplus \ldots \oplus \mathbb{Z}/p^{r_s}$ for some $r_1, \ldots, r_s$ and some s with $r_1 + \ldots + r_s = mn$. Since $\Theta_m(p) = \{a \in \overline{Q(E^0)} \mid [p](a) = 0\} = \Theta_1$ we get $|\Theta_m(p)| = p^n$ and it follows that $s = n$. The fact that $[p^m](a) = 0$ for all $a \in \Theta_m$ gives $r_i \leq m$ for each i. Thus we conclude that $r_i = m$ for all i and $\Theta_m \simeq (\mathbb{Z}/p^m)^n$. Choosing compatible isomorphisms for each m then gives an isomorphism $\Theta \simeq (\mathbb{Z}/p^\infty)^n$. $\square$

We now outline the development of the generalised character map for $E^0(BG)$ for finite groups G.

Definition 5.2. Let $A \subseteq \overline{Q(E^0)}$ be a finite abelian group under $+_F$. Then we define $L_A \subseteq \overline{Q(E^0)}$ to be the ring generated by $\mathbb{Q} \otimes E^0$ and A. Note that given a map of two such groups $f : A \to B$ there is an induced $\mathbb{Q} \otimes E^0$-algebra map $f_* : L_A \to L_B$ sending each $a \in A \subseteq L_A$ to $f(a) \in B \subseteq L_B$. We define $L_m = L_{\Theta_m}$.

Proposition 5.3. *Let $A \subseteq \overline{Q(E^0)}$ be a finite abelian p-group under $+_F$. Then there is a unique E^0-algebra map $\psi_A : E^0(BA^*) \to L_A$ such that $\psi_A(a^*(x)) = a$ for all $a \in A = (A^*)^* = Hom_{cts}(A^*, S^1)$. Further these maps are natural for homomorphisms of such groups.*

Proof. Choose an isomorphism $A \xrightarrow{\sim} \prod_{i=1}^m C_{p^{r_i}}$ to get generators $a_1, \ldots, a_m$ for A. Then, by Lemma 4.51, we get $E^0(BA^*) \simeq E^0[\![x_1, \ldots, x_m]\!]/([p^{r_1}](x_1), \ldots, [p^{r_m}](x_m))$ where, viewing a_i as an element of $(A^*)^*$, we have $x_i = {a_i}^*(x)$. Hence, since $[p^{r_i}](a_i) = 0$ in L_A, there is an E^0-algebra map $\psi_A : E^0(BA^*) \to L_A$ given by ${a_i}^*(x) = x_i \mapsto a_i$.

Now, for any $a, b \in A = (A^*)^*$, the commutative diagram

$$\begin{tikzcd} A^* \arrow[r, "a\times b"] \arrow[dr, "a+b"'] & S^1 \times S^1 \arrow[d, "\mu"] \\ & S^1 \end{tikzcd} \qquad \text{induces} \qquad \begin{tikzcd} E^0(BA^*) & E^0[\![x_{(1)}, x_{(2)}]\!] \arrow[l, "(a\times b)^*"'] \\ & E^0[\![x]\!] \arrow[u, "\mu^*"'] \arrow[ul, "(a+b)^*"] \end{tikzcd}$$

which translates to $(a+b)^*(x) = F((a \times b)^*(x_{(1)}), (a \times b)^*(x_{(2)})) = a^*(x) +_F b^*(x)$. Hence, given an arbitrary $a \in A$ we can write $a = \sum_i a_{j_i}$ to get

$$\psi_A(a^*(x)) = \psi_A((\textstyle\sum_i a_{j_i})^*(x)) = \psi_A(\textstyle\sum_F x_{j_i}) = \textstyle\sum_F a_{j_i} = a.$$

Thus the map ψ_A has the required property and is unique by construction. For naturality, let $B \subseteq \overline{Q(E^0)}$ be another finite abelian group under $+_F$ and $f : A \to B$ a group homomorphism. Then we have maps $f_* : E^0(BA^*) \to E^0(BB^*)$ and $f_* : L_A \to L_B$ and

$$\psi_B(f_*(a^*(x))) = \psi_B(f(a)_*(x)) = f(a) = f_*(a) = f_*(\psi_A(a^*(x)))$$

which is the claimed naturality condition. $\square$

This gives us the following immediate corollary.

Corollary 5.4. *The inclusions $\Theta_m \hookrightarrow \Theta_{m+1}$ induce commutative squares*

$$\begin{tikzcd} E^0(B(\Theta_m)^*) \arrow[r, "\psi_m"] \arrow[d] & L_m \arrow[d, hook] \\ E^0(B(\Theta_{m+1})^*) \arrow[r, "\psi_{m+1}"'] & L_{m+1}. \end{tikzcd}$$

Hence, writing $L = \bigcup_m L_m$ we get an induced map $\psi : \varinjlim E^0(B(\Theta_m)^*) \to L$.

Definition 5.5. Given topological groups G and H we have an action of G on $\mathrm{Hom}_{\mathrm{cts}}(H, G)$ given by $(g.\alpha)(h) = g\alpha(h)g^{-1}$ for $g \in G, \alpha \in \mathrm{Hom}_{\mathrm{cts}}(H, G)$ and $h \in H$. We define the set $\mathrm{Rep}(H, G)$ by $\mathrm{Rep}(H, G) = \mathrm{Hom}_{\mathrm{cts}}(H, G)/G$.

Remark 5.6. Notice that, by Proposition 4.63, if $\alpha \in \mathrm{Hom}_{\mathrm{cts}}(H, G)$ then the induced map $\alpha^* : E^0(BG) \to E^0(BH)$ depends only on the class of α in $\mathrm{Hom}_{\mathrm{cts}}(H, G)/G = \mathrm{Rep}(H, G)$. Also note that if G is a finite group, then any homomorphism $f : \Theta^* \to G$ must factor through $\Theta^*/\ker f$ which is a finite quotient of Θ^* and hence discrete. It follows that f is automatically continuous; that is, $\mathrm{Hom}_{\mathrm{cts}}(\Theta^*, G) = \mathrm{Hom}(\Theta^*, G)$.

Definition 5.7. Let G be a finite group and let $\alpha \in \mathrm{Hom}(\Theta^*, G)$. Then α induces maps $\alpha_m^* : E^0(BG) \to E^0(B\Theta_m^*)$ for each m which fit into the diagram

$$\begin{tikzcd} E^0(BG) \arrow[r, "\alpha_m^*"] \arrow[dr, "\alpha_{m+1}^*"'] & E^0(B\Theta_m^*) \arrow[d] \\ & E^0(B\Theta_{m+1}^*) \end{tikzcd}$$

and hence a map $\alpha^* : E^0(BG) \to \varinjlim E^0(B\Theta_m^*)$. We define χ_α to be the composite

$$E^0(BG) \xrightarrow{\alpha^*} \varinjlim E^0(B\Theta_m^*) \xrightarrow{\psi} L.$$

Finally, writing $\mathrm{Map}(S, T)$ for the set of functions (that is, set-maps) $S \to T$, we define $\chi : E^0(BG) \to \mathrm{Map}(\mathrm{Rep}(\Theta^*, G), L)$ by $(\chi(a))(\alpha) = \chi_\alpha(a)$ for $a \in E^0(BG)$ and $\alpha \in \mathrm{Hom}(\Theta^*, G)$ and refer to it as the *generalised character map for G*. By the remarks above this is a well defined map of E^0-algebras.

We are now able to state the result of Hopkins, Kuhn and Ravenel.

Proposition 5.8. *(Hopkins, Kuhn, Ravenel) Let G be a finite group. The map χ defined above induces an isomorphism of L-algebras*

$$L \otimes_{E^0} E^0(BG) \xrightarrow{\sim} \mathit{Map}(\mathit{Rep}(\Theta^*, G), L).$$

Proof. This is part of Theorem C from [HKR00]. $\square$

5.2 Generalised characters and the finite general linear groups

In this section we apply the generalised character theory to the group $GL_d(\mathbb{F}_q)$, where $q = l^r$ is a power of some prime different to p and d is a positive integer. Our main result is the following.

Theorem A. *Let $\Phi = (\mathbb{Z}/p^\infty)^n$ and let Λ be the subgroup of $\mathbb{Z}_p^\times$ generated by q. Then there is a bijection $Rep(\Theta^*, GL_d(\mathbb{F}_q)) \xrightarrow{\sim} (\Phi^d/\Sigma_d)^\Lambda$. Further, there there is a natural addition structure on each of $\coprod_k Rep(\Theta^*, GL_k(\mathbb{F}_q))$ and $\coprod_k (\Phi^k/\Sigma_k)^\Lambda$ and the bijection*

$$\coprod_k Rep(\Theta^*, GL_k(\mathbb{F}_q)) \xrightarrow{\sim} \coprod_k (\Phi^k/\Sigma_k)^\Lambda$$

is an isomorphism of abelian semigroups.

We will study two particular cases, firstly where $d < p$ and secondly where $d = p$ and find that $(\Phi^d/\Sigma_d)^\Lambda$ is easy to understand in both cases. Further, this will give us a good understanding of the ring $L \otimes_{E^0} E^0(BGL_p(\mathbb{F}_q))$.

5.2.1 Representation theory

As above, $q = l^r$ is a power of a prime different to p and d is a positive integer. We may write $\overline{\mathbb{F}}_q$ for the algebraic closure $\overline{\mathbb{F}}_l$ and $\mathbb{F}_{q^d}$ for $\mathbb{F}_{l^{rd}}$.

Definition 5.9. We write $\Phi = \mathrm{Rep}(\Theta^*, GL_1(\overline{\mathbb{F}}_l)) = \mathrm{Hom}_{\mathrm{cts}}(\Theta^*, \overline{\mathbb{F}}_l^\times)$.

As in Section 2.1.4, let $\Gamma = \langle F_q \rangle$ be the subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_q)$ generated by the r^{th} power of the Frobenius map. Since Γ acts on $\overline{\mathbb{F}}_l$ we have an induced action of Γ on Φ given by $(\gamma.\alpha)(a) = \gamma.\alpha(a)$ (for $\gamma \in \Gamma$, $\alpha : \Theta^* \to GL_1(\overline{\mathbb{F}}_l)$ and $a \in \Theta^*$). Note also that we have a componentwise action of Γ on $GL_d(\overline{\mathbb{F}}_l)$ which induces an action of Γ on $\mathrm{Rep}(\Theta^*, GL_d(\overline{\mathbb{F}}_l))$ in the obvious way.

Proposition 5.10. *Point-wise multiplication makes Φ into an abelian p-group. Further, there is an isomorphism $\Phi \simeq (\mathbb{Z}/p^\infty)^n$.*

Proof. Let $\phi \in \Phi$. Then $\phi : \Theta^* \to \overline{\mathbb{F}}_l^\times$ is continuous. In particular, since $\overline{\mathbb{F}}_l^\times$ is discrete, $\ker(\phi) = \phi^{-1}(1)$ is open. But the open neighbourhoods of 0 in $\mathbb{Z}_p$ are just the subgroups $p^m\mathbb{Z}_p$ for $m \in \mathbb{N}$. Since $\Theta^* \simeq \mathbb{Z}_p^n$ it follows that $\ker(\phi) \supseteq p^N\Theta^*$ for some N. Thus $\phi^{p^N}(a) = \phi(a)^{p^N} = \phi(p^N a) = 0$ and ϕ has order a power of p.

For the final statement we notice that

$$\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \overline{\mathbb{F}}_l^\times) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathrm{Syl}_p(\overline{\mathbb{F}}_l^\times)) \simeq \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}/p^\infty).$$

Since any continuous homomorphism $f : \mathbb{Z}_p \to \mathbb{Z}/p^\infty$ will have finite image it will be determined by $f(1)$. Thus $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}/p^\infty) = \mathbb{Z}/p^\infty$ and the result follows. $\square$

Let K be a field. Then, remembering that any two representations $\rho_1, \rho_2 : \Theta^* \to GL_d(K)$ are isomorphic if and only if there is some $g \in GL_d(K)$ with $\rho_1 = g\rho_2 g^{-1}$, we see that there is an obvious correspondence of $\mathrm{Rep}(\Theta^*, GL_d(K))$ with isomorphism classes of continuous

d-dimensional K-representations of Θ^*. We will denote elements in the former by $[\alpha]$ where α is a homomorphism from Θ^* to $GL_d(K)$ and elements of the latter by pairs (V, α) where V is a d-dimensional K-vector space and $\alpha : \Theta^* \to GL(V)$.

Definition 5.11. Using the correspondence outlined above we define $\mathrm{Irr}(\Theta^*, GL_d(K))$ to be the subset of $\mathrm{Rep}(\Theta^*, GL_d(K))$ corresponding to the irreducible representations.

Our next step is to try to understand the $\overline{\mathbb{F}}_l$-representation theory of Θ^*. For this we need the following results. Recall that Schur's Lemma (see [Ser77]) tells us that any map $f : V \to W$ of irreducible K-representations of a group G is either zero or an isomorphism.

Lemma 5.12. *Let G be an abelian group and K an algebraically closed field. Then any irreducible finite dimensional K-representation of G is one-dimensional.*

Proof. Let (V, ρ) be any irreducible finite-dimensional K-representation of G. If $g \in G$ then, since K is algebraically closed, $\rho(g)$ has an eigenvalue, λ say. Thus $\ker(\rho(g) - \lambda.\mathrm{id}_V) \neq \emptyset$ since it contains the eigenvectors corresponding to λ. Hence, by Schur's lemma, we must have $\rho(g) - \lambda.\mathrm{id}_V = 0$ so that $\rho(g) = \lambda.\mathrm{id}_V$. Given any subspace W of V we then get $\rho(g)(W) \subseteq W$ so that W is a subrepresentation of V. But V is irreducible, so that either $W = 0$ or $W = V$. Hence V has no non-trivial proper subspaces and so is one-dimensional, as required. $\square$

Lemma 5.13. *Let K be a (discrete) field with p invertible in K. Then every continuous finite-dimensional K-representation of Θ^* is a sum of irreducible representations.*

Proof. This is essentially Maschke's theorem. Let (V, ρ) be a continuous finite-dimensional K-representation of Θ^*. Suppose $\dim V = d > 1$. Note that, as in the proof of Proposition 5.10, $\ker(\rho) \supseteq p^N\Theta^*$ for some $N \in \mathbb{N}$ and ρ factors through the finite abelian p-group $G = \Theta^*/p^N\Theta^*$. If V is irreducible, then we are done. Otherwise there is a proper subrepresentation W of V of dimension less than d. Let W' be any vector space complement of W in V. Writing π for the projection $V = W' \oplus W \twoheadrightarrow W$ we define a map $r : V \to V$ by

$$r(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi(\rho(g)^{-1}(v)).$$

Then the image of ρ is contained in W and for any $w \in W$ we have $r(w) = w$; that is, r is a projection of V onto W. Let $U = \ker r$ so that $V = U \oplus W$. For any $h \in G$ it is easy to check that we have $r(\rho(h)v) = \rho(h)r(v)$ so that if $u \in U$ we have $r(\rho(h)u) = \rho(h)r(u) = 0$ showing that $\rho(h)u \in U$. Thus U is also a subrepresentation of V and $V = U \oplus W$ is a sum of representations of Θ^* of dimension less than d. The result then follows by induction, noting that all 1-dimensional representations are (trivially) irreducible. $\square$

Corollary 5.14. *Every continuous $\overline{\mathbb{F}}_l$-representation of Θ^* is a sum of 1-dimensional representations. Moreover, this decomposition is unique up to reordering.*

Proof. The first statement is immediate from Lemmas 5.12 and 5.13. The second is a standard application of Schur's Lemma: any irreducible representation W appears in the decomposition for V with multiplicity $\dim \mathrm{Hom}(V, W)$. $\square$

Proposition 5.15. *Let Σ_d act on Φ^d by permuting the factors. Then the map $\Phi^d \to$ Rep$(\Theta^*, GL_d(\overline{\mathbb{F}}_l))$ given by $(\phi_1, \ldots, \phi_d) \mapsto [\phi_1 \oplus \ldots \oplus \phi_d]$ induces a Γ-equivariant bijection $\Phi^d/\Sigma_d \xrightarrow{\sim}$ Rep$(\Theta^*, GL_d(\overline{\mathbb{F}}_l))$.*

Proof. Apply Corollary 5.14 to see that the map is a bijection. For $\gamma \in \Gamma$ we have

$$\gamma.[\phi_1, \ldots, \phi_d] = [\gamma.\phi_1, \ldots, \gamma.\phi_d] \mapsto [\gamma.\phi_1 \oplus \ldots \oplus \gamma.\phi_d] = \gamma.[\phi_1 \oplus \ldots \oplus \phi_d]$$

so the map is Γ-equivariant. $\square$

Lemma 5.16. *Let V be an $\mathbb{F}_q$-representation of Θ^* of dimension d. Then V is irreducible if and only if there is a 1-dimensional $\mathbb{F}_{q^d}$-vector space structure on V and a representation $\phi : \Theta^* \to GL_1(\mathbb{F}_{q^d})$ with $\min\{s \in \mathbb{N} \mid \phi^{q^s} = \phi\} = d$ such that the diagram*

$$\begin{tikzcd} \Theta^* \arrow[r] \arrow[d, "\phi"'] & \mathbb{F}_q[\Theta^*] \arrow[r] \arrow[d] & End_{\mathbb{F}_q}(V) \\ \mathbb{F}_{q^d}^{\times} \arrow[r] & \mathbb{F}_{q^d} \arrow[r] & End_{\mathbb{F}_{q^d}}(V) \arrow[u] \end{tikzcd}$$

commutes.

Proof. Suppose V is irreducible. Viewing V as an $\mathbb{F}_q[\Theta^*]$-module, let $\alpha : \mathbb{F}_q[\Theta^*] \to \mathrm{End}_{\mathbb{F}_q}(V)$ be the map corresponding to the action of $\mathbb{F}_q[\Theta^*]$ on V and put $K = \mathrm{image}(\alpha)$. We will show that $K \simeq \mathbb{F}_{q^d}$ and that V is a 1-dimensional K-vector space. Let $0 \neq \psi \in K$ and write $\psi = \alpha(a)$ for $a \in \mathbb{F}_q[\Theta^*]$. Then, since Θ^* and hence $\mathbb{F}_q[\Theta^*]$ is commutative, taking any $b \in \mathbb{F}_q[\Theta^*]$ we have

$$\psi(b.v) = (\alpha(a) \circ \alpha(b))(v) = (\alpha(ab))(v) = (\alpha(ba))(v) = \alpha(b)(\psi(v)) = b.\psi(v).$$

Thus ψ is an $\mathbb{F}_q[\Theta^*]$-endomorphism of V and hence, by Schur's lemma, is an automorphism. Further, by the Cayley-Hamilton theorem, ψ satisfies an equation $\psi^r + c_1\psi^{r-1} + \ldots + c_r = 0$ for some $c_1, \ldots, c_r \in \mathbb{F}_q$ with $c_r = \det(\psi) \in \mathbb{F}_q^{\times}$. Then $\psi^{-1} = -\alpha((c_{r-1} + \ldots + c_1 a^{r-2} + a^{r-1})c_r^{-1})$ lies in K and K is a field.

Notice that, by definition of K, V is a K-vector space. Suppose $\dim_K(V) > 1$ and let U be a proper K-subspace of V. Then, for $a \in \mathbb{F}_q[\Theta^*]$ we have $\alpha(a) \in K$ so $\alpha(a)(u) \in U$ for all $u \in U$. Thus U is an $\mathbb{F}_q[\Theta^*]$-module; that is, U is a proper subrepresentation of V contradicting the fact that V is irreducible. Hence we must have $\dim_K(V) = 1$ so that $V \simeq K$. Since $\dim_{\mathbb{F}_q}(V) = d$ it follows that $V \simeq K \simeq \mathbb{F}_{q^d}$, as required.

For the final statement, let ϕ be the composition $\Theta^* \to \mathbb{F}_q[\Theta^*] \xrightarrow{\alpha} \mathrm{End}_{\mathbb{F}_q}(V)$. Then, since $\mathrm{image}(\phi) \subseteq \mathbb{F}_{q^d}$, it follows that $\phi^{q^d} = \phi$. Suppose that $\phi^{q^s} = \phi$ for some $s < d$. Then $\mathrm{image}(\phi) \subseteq \mathbb{F}_{q^s} \subset \mathbb{F}_{q^d}$. But then $\mathbb{F}_{q^s}$ is a proper $\mathbb{F}_q$-subrepresentation of V, contradicting the fact that V is irreducible over $\mathbb{F}_q$.

Conversely, suppose we have an $\mathbb{F}_{q^d}$-representation $\phi : \Theta^* \to GL_1(\mathbb{F}_{q^d}) = \mathbb{F}_{q^d}^{\times}$ satisfying the relevant properties. Note that ϕ factors through a finite p-group $\Theta^*/p^N\Theta^*$ for some N, so that $\mathrm{image}(\phi) \subseteq \mathrm{Syl}_p(\mathbb{F}_{q^d}) \simeq C_{p^v}$, where $v = v_p(q^d - 1)$. Further, the conditions on ϕ ensure that ϕ has order precisely p^v and that, for any $s < d$, writing $v_s = v_p(q^s - 1)$ we have $v_s < v$. Suppose that $(\mathbb{F}_{q^d}, \phi)$ is not irreducible and write $\mathbb{F}_{q^d} \simeq U_1 \oplus \ldots \oplus U_m$ as a sum of irreducible $\mathbb{F}_q$-representations U_i of dimensions d_i. Applying the first part of the result we find that $U_i \simeq \mathbb{F}_{q^{d_i}}$ with action map $\phi_i : \Theta^* \to (\mathbb{F}_{q^{d_i}})^{\times}$. Then we find that $\mathrm{image}(\phi_i) \simeq C_{p^{v_{d_i}}}$. In particular, since $v_i < v$, we find $(\phi_i)^{p^{v-1}} = 1$ so that $(\phi_1 \times \ldots \times \phi_m)^{p^{v-1}} = 1$. Hence we see that $\phi^{p^{v-1}} = 1$, contradicting the assumption on its order. Hence $(\mathbb{F}_{q^d}, \phi)$ is irreducible. $\square$

We move towards an understanding of $\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q))$. By post-composing with the inclusion $GL_d(\mathbb{F}_q) \rightarrowtail GL_d(\overline{\mathbb{F}}_l)$ we get a map $\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q)) \to \mathrm{Rep}(\Theta^*, GL_d(\overline{\mathbb{F}}_l))$. Since $\mathbb{F}_q = \overline{\mathbb{F}}_l^{\Gamma}$ we see that this map lands in the Γ-invariants. Thus, using Γ-equivariance of the bijection $\Phi^d/\Sigma_d \xrightarrow{\sim} \mathrm{Rep}(\Theta^*, GL_d(\overline{\mathbb{F}}_l))$, we find that we have a map $\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q)) \to (\Phi^d/\Sigma_d)^\Gamma$. We will shortly show this is bijective, but first we need a result from field theory.

Lemma 5.17. *Let K be a field and L be a Galois extension of K with Galois group G. Then there is a $\overline{K}$-vector space isomorphism $\overline{K} \otimes_K L \xrightarrow{\sim} Map(G, \overline{K})$ given by $a \otimes b \mapsto (g \mapsto a.g(b))$.*

Proof. This is a well known result. An application of [Ada81, Theorem 14.1] shows that the map $\overline{K}[G] \to \mathrm{Hom}_K(L, \overline{K})$ is an isomorphism of $\overline{K}$-vector spaces and the result follows on applying $\mathrm{Hom}_{\overline{K}}(-, \overline{K})$. □

Proposition 5.18. *Let V be an irreducible $\mathbb{F}_q$-representation of Θ^* of dimension d. Then there is a continuous $\overline{\mathbb{F}}_l$-representation $(\overline{\mathbb{F}}_l, \phi)$ of Θ^* such that $\min\{s \in \mathbb{N} \mid \phi^{l^{rs}} = \phi\} = d$ and*

$$\overline{\mathbb{F}}_l \otimes_{\mathbb{F}_q} V \simeq (\overline{\mathbb{F}}_l, \phi) \oplus (\overline{\mathbb{F}}_l, \phi^{l^r}) \oplus \ldots \oplus (\overline{\mathbb{F}}_l, \phi^{l^{r(d-1)}}).$$

Proof. Using Lemma 5.16 we can assume that $V = \mathbb{F}_{l^{rd}}$ and we get an $\overline{\mathbb{F}}_l$-representation $\phi : \Theta^* \to GL_1(\mathbb{F}_{l^{rd}}) \rightarrowtail GL_1(\overline{\mathbb{F}}_l)$ with the required properties on its order. Next we apply Lemma 5.17 to the extension $\mathbb{F}_{l^{rd}}/\mathbb{F}_q$. Note that $\mathrm{Gal}(\mathbb{F}_{l^{rd}}/\mathbb{F}_q) = \langle F^r \rangle$ is cyclic of order d generated by the r^{th} power of the Frobenius map so that $\mathrm{Map}(\mathrm{Gal}(\mathbb{F}_{l^{rd}}/\mathbb{F}_q), \overline{\mathbb{F}}_l)$ is just $\overline{\mathbb{F}}_l^d$. Thus we have an isomorphism of $\overline{\mathbb{F}}_l$-vector spaces $\overline{\mathbb{F}}_l \otimes_{\mathbb{F}_q} \mathbb{F}_{l^{rd}} \xrightarrow{\sim} \overline{\mathbb{F}}_l^d$ given by $a \otimes b \mapsto (ab, ab^{l^r}, \ldots, ab^{l^{r(d-1)}})$. It is clear that, giving the right-hand side a Θ^*-action by $x.(a_1, \ldots, a_d) = (a_1\phi(x), a_2\phi(x)^{l^r}, \ldots, a_d\phi(x)^{l^{r(d-1)}})$, the isomorphism is Θ^*-equivariant and thus is the claimed isomorphism of $\overline{\mathbb{F}}_l$-representations. □

We will call an element of $(\Phi^d/\Sigma_d)^\Gamma$ *irreducible* if it is of the form $[\phi, \phi^{l^r}, \ldots, \phi^{l^{r(d-1)}}]$ for some $\phi \in \Phi$ with $\min\{s \in \mathbb{N} \mid \phi^{l^{rs}} = \phi\} = d$. We write $\mathrm{Irr}_d(\Phi)$ for the set of irreducible elements of $(\Phi^d/\Sigma_d)^\Gamma$.

Proposition 5.19. *The binary operation $+ : (\Phi^s/\Sigma_s)^\Gamma \times (\Phi^t/\Sigma_t)^\Gamma \to (\Phi^{s+t}/\Sigma_{s+t})^\Gamma$ given by $[\phi_1, \ldots, \phi_s] + [\phi'_1, \ldots, \phi'_t] = [\phi_1, \ldots, \phi_s, \phi'_1, \ldots, \phi'_t]$ makes $\coprod_{d>0}(\Phi^d/\Sigma_d)^\Gamma$ into an abelian semigroup freely generated by the irreducible elements.*

Proof. If $[\phi_1, \ldots, \phi_d] \in \Phi^d/\Sigma_d$ is Γ-invariant then $[\phi_1, \ldots, \phi_d] = [F^r.\phi_1, \ldots, F^r.\phi_d]$ so that for each i there is a j with $\phi_j = \phi_i^{l^r}$. Thus $\phi_1, \phi_1^{l^r}, \phi_1^{l^{2r}}, \ldots$ all appear in the expression $[\phi_1, \ldots, \phi_d]$ and so, by finiteness, we must at some point have $\phi_1^{l^{rs}} = \phi_1$. Thus $[\phi_1, \ldots, \phi_1^{l^{r(s-1)}}]$ is an irreducible element of $(\Phi^s/\Sigma_s)^\Gamma$ which appears as a summand in $[\phi_1, \ldots, \phi_d]$. Continuing in this way it is easy to see that $[\phi_1, \ldots, \phi_d]$ decomposes as a sum of irreducibles in a unique way. □

Proposition 5.20. *The map $\alpha : Rep(\Theta^*, GL_d(\mathbb{F}_q)) \to (\Phi^d/\Sigma_d)^\Gamma$ is bijective.*

Proof. By Proposition 5.18, α restricts to a map $\mathrm{Irr}(\Theta^*, GL_d(\mathbb{F}_q)) \to \mathrm{Irr}_d(\Phi)$; we will show that this is a bijection. It then follows, using Lemma 5.13, that $\coprod_{d>0} \mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q))$ bijects with $\coprod_{d>0}(\Phi^d/\Sigma_d)^\Gamma$ and from that we conclude that α itself is bijective.

Take any irreducibles $V, W \in \mathrm{Irr}(\Theta^*, GL_d(\mathbb{F}_q))$ with $\alpha(V) = \alpha(W)$. Then, as before, we can assume that $V = (\mathbb{F}_{l^{rd}}, \phi)$ and $W = (\mathbb{F}_{l^{rd}}, \phi')$ for some $\phi, \phi' : \Theta^* \to \mathbb{F}_{l^{rd}}^\times$. Viewing ϕ and ϕ' as maps $\Theta^* \to \mathbb{F}_{l^{rd}}^\times \to \overline{\mathbb{F}}_l^\times$ we have $\alpha(V) = [\phi, \phi^{l^r}, \ldots, \phi^{l^{r(d-1)}}]$ and $\alpha(W) = [\phi', (\phi')^{l^r}, \ldots, (\phi')^{l^{r(d-1)}}]$. Thus we must have $\phi' = \phi^{l^{rs}}$ for some $0 \le s < d$ and it follows that the $\mathbb{F}_q$-linear isomorphism $V \xrightarrow{\sim} W$, $a \mapsto a^{l^{rs}}$ gives an isomorphism of $\mathbb{F}_q$-representations $V \simeq W$; that is $V = W$ in $\mathrm{Irr}(\Theta^*, GL_d(\mathbb{F}_q))$.

For surjectivity, given $[\phi, \phi^{l^r}, \ldots, \phi^{l^{r(d-1)}}] \in \mathrm{Irr}_d(\Phi)$ we have $\phi^{l^{rd}} = \phi$ so that the image of ϕ is contained in $\mathbb{F}_{l^{rd}} \subseteq \overline{\mathbb{F}}_l$. Using Proposition 5.16 the irreducible d-dimensional $\mathbb{F}_q$-representation $V = (\mathbb{F}_{l^{rd}}, \phi)$ satisfies $\alpha(V) = [\phi, \phi^{l^r}, \ldots, \phi^{l^{r(d-1)}}]$. ☐

Hence, to get a good understanding of $\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q))$ it suffices to study the set $(\Phi^d/\Sigma_d)^\Gamma$.

Lemma 5.21. *The set $Irr_d(\Phi)$ bijects with Γ-orbits of size d on Φ.*

Proof. Each of these consists of all unordered d-tuples $(\phi, \phi^q, \ldots, \phi^{q^{d-1}})$ with $\phi^{q^d} = \phi$. ☐

To take our analysis further, we will make the simplifying assumption that $v_p(q-1) = v > 0$.

Proposition 5.22. *Suppose that $v_p(q-1) = v > 0$. Then*

$$Irr_d(\Phi) = \begin{cases} \Phi(p^v) & \text{if } d = 1 \\ (\Phi(p^{v+k}) \setminus \Phi(p^{v+k-1}))/\Gamma & \text{if } d = p^k \text{ for some } k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $d = 1$ is clear. For $d > 1$, using the fact that Φ is a p-group we see that $\phi \in \Phi$ generates a Γ-orbit of size d if and only if it has order $p^{v_p(q^d-1)}$ and $v_p(q^s - 1) < v_p(q^d - 1)$ for all $s < d$. By Proposition 2.33, this latter requirement is equivalent to $v_p(s) < v_p(d)$ for all $s < d$ which is satisfied precisely when d is a power of p. ☐

We look at two easy to understand cases.

Proposition 5.23. *Suppose that $v_p(q-1) = v > 0$ and $d < p$. Then*

$$(\Phi^d/\Sigma_d)^\Gamma = Irr_1(\Phi)^d/\Sigma_d = \Phi(p^v)^d/\Sigma_d.$$

Proof. Since every element of the left-hand side is a sum or irreducibles and $d < p$ it follows from Proposition 5.22 that all these irreducibles must be in $\mathrm{Irr}_1(\Phi)$. ☐

Remark 5.24. Suppose that $v_p(q-1) = v > 0$ and $d < p$. As $\Phi(p^v)$ consists of all $\phi \in \Phi$ for which $\phi^{p^v} = 1$, any such ϕ therefore satisfies $\phi^q = \phi$ and thus has image contained in $\mathbb{F}_q$. Hence $\mathrm{Rep}(\Theta^*, GL_1(\mathbb{F}_q)) \to \Phi(p^v)$ is a bijection and the diagram

$$\begin{tikzcd} \mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q)) \arrow[r, "\sim"] & (\Phi^d/\Sigma_d)^\Gamma \\ \mathrm{Rep}(\Theta^*, GL_1(\mathbb{F}_q))^d/\Sigma_d \arrow[r, "\sim"] \arrow[u] & \Phi(p^v)^d/\Sigma_d \arrow[u, "\wr"'] \end{tikzcd}$$

shows that $\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q)) = \mathrm{Rep}(\Theta^*, GL_1(\mathbb{F}_q))^d/\Sigma_d$; that is, any representation $\rho : \Theta^* \to GL_d(\mathbb{F}_q)$ is diagonalisable.

Proposition 5.25. *Suppose that $v_p(q-1) = v > 0$. Then*

$$(\Phi^p/\Sigma_p)^\Gamma = (Irr_1(\Phi)^p/\Sigma_p) \coprod Irr_p(\Phi) = (\Phi(p^v)^p/\Sigma_p) \coprod (\Phi(p^{v+1}) \setminus \Phi(p^v))/\Gamma.$$

Proof. Writing any element in the left-hand side as a sum of irreducibles, it is clear that either it is already irreducible or it is a sum of 1-dimensional irreducibles. ☐

For $d > p$ things get more complicated.

5.2.2 Applying the character theory

Now that we have a good understanding of the set $\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q))$ we can apply the generalised character theory to the group $GL_d(\mathbb{F}_q)$ to better understand the structure of its cohomology. We first need the following lemma.

Lemma 5.26. *Let G act on a set X. Then, if Y is any set, G acts on $Map(X, Y)$ by $(g.f)(x) = f(g^{-1}.x)$ and the obvious map $Map(X/G, Y) \to Map(X, Y)$ gives a bijection*

$$Map(X/G, Y) \simeq Map(X, Y)^G.$$

Proof. Given $f : X/G \to Y$ it is clear that the map $\tilde{f} : X \to Y$, $x \mapsto f(\bar{x})$ is G-invariant and it is easy to check that this construction gives a bijection of sets. $\square$

Recall that T_d denotes the maximal torus of $GL_d(\mathbb{F}_q)$.

Proposition 5.27. *Suppose that $v_p(q-1) = v > 0$ and $d < p$. Then the restriction map induces an isomorphism $L \otimes_{E^0} E^0(BGL_d(\mathbb{F}_q)) \xrightarrow{\sim} L \otimes_{E^0} E^0(BT_d)^{\Sigma_d}$.*

Proof. Note first that $\mathrm{Rep}(\Theta^*, T_d) = \mathrm{Rep}(\Theta^*, (\mathbb{F}_q^\times)^d) = \mathrm{Rep}(\Theta^*, \mathbb{F}_q^\times)^d = \mathrm{Rep}(\Theta^*, GL_1(\mathbb{F}_q))^d$. Then, using the remarks in 5.24, applying the generalised character theory we get a diagram

$$\begin{tikzcd}
L \otimes_{E^0} E^0(BGL_d(\mathbb{F}_q)) \arrow[r, "\sim"] \arrow[d] & \mathrm{Map}(\mathrm{Rep}(\Theta^*, GL_d(\mathbb{F}_q)), L) \arrow[d, "\wr"] \\
L \otimes_{E^0} E^0(BT_d)^{\Sigma_d} \arrow[r, "\sim"] \arrow[d] & \mathrm{Map}(\mathrm{Rep}(\Theta^*, GL_1(\mathbb{F}_q))^d, L)^{\Sigma_d} \arrow[d] \\
L \otimes_{E^0} E^0(BT_d) \arrow[r, "\sim"] & \mathrm{Map}(\mathrm{Rep}(\Theta^*, GL_1(\mathbb{F}_q))^d, L)
\end{tikzcd}$$

which gives the result. $\square$

In fact it is not hard to show that the map $E^0(BGL_d(\mathbb{F}_q)) \to E^0(BT_d)^{\Sigma_d}$ is itself an isomorphism, which we do in Chapter 6.

We next look at the case where $d = p$ and again take the simplifying assumption that $v_p(q-1) = v > 0$. Fix a basis for $\mathbb{F}_{q^p}$ as a vector space over $\mathbb{F}_q$. Then for any $a \in \mathbb{F}_{q^p}^\times$ we have an $\mathbb{F}_q$-linear automorphism $\mu_a : \mathbb{F}_{q^p} \to \mathbb{F}_{q^p}$ given by multiplication by a. Let $\mu : \mathbb{F}_{q^p}^\times \to GL_p(\mathbb{F}_q), a \mapsto \mu_a$ be the corresponding group homomorphism. This gives us a map $\mu^* : E^0(BGL_p(\mathbb{F}_q)) \to E^0(B\mathbb{F}_{q^p}^\times)$. We need the following lemma.

Lemma 5.28. *For any $s > 0$, the quotient maps $E^0[\![x]\!]/[p^{s+1}](x) \to E^0[\![x]\!]/[p^s](x)$ and $E^0[\![x]\!]/[p^{s+1}](x) \to E^0[\![x]\!]/\langle p\rangle([p^s](x))$ induce an isomorphism*

$$\mathbb{Q} \otimes \frac{E^0[\![x]\!]}{[p^{s+1}](x)} \xrightarrow{\sim} \left(\mathbb{Q} \otimes \frac{E^0[\![x]\!]}{[p^s](x)}\right) \times \left(\mathbb{Q} \otimes \frac{E^0[\![x]\!]}{\langle p\rangle([p^s](x))}\right)$$

where $\langle p\rangle(t) = [p](t)/t \in E^0[\![t]\!]$ is the divided p-series.

Proof. Recall that $[p^{s+1}](x)$ and $[p^s](x)$ are unit multiples of the Weierstrass polynomials $g_{s+1}(x)$ and $g_s(x)$ respectively. Further $g_s(x)$ divides $g_{s+1}(x)$. Put $g(x) = g_{s+1}(x)/g_s(x)$. Then, since $\langle p\rangle(x) = p \bmod x$ it follows that $\langle p\rangle([p^s](x)) = p \bmod [p^s](x)$ so that p is in the ideal of $E^0[\![x]\!]/[p^{s+1}](x)$ generated by $g(x)$ and $g_s(x)$. Hence, using the Chinese remainder theorem (in particular, Corollary 2.22) the result follows. $\square$

Note that $v_p(|\mathbb{F}_{q^p}^\times|) = v_p(q^p - 1) = v + 1$ and our complex orientation gives us a presentation $E^0(B\mathbb{F}_{q^p}^\times) \simeq E^0[\![x]\!]/[p^{v+1}](x)$. Further, Γ acts on $E^0(B\mathbb{F}_{q^p}^\times)$ and hence also on the quotient ring $E^0[\![x]\!]/\langle p\rangle([p^v](x))$. Write $q : E^0(B\mathbb{F}_{q^p}^\times) \to E^0[\![x]\!]/\langle p\rangle([p^v](x))$ for the quotient map and let $\alpha = q \circ \mu^* : E^0(BGL_p(\mathbb{F}_q)) \to E^0[\![x]\!]/\langle p\rangle([p^v](x))$.

Proposition 5.29. *Suppose that $v_p(q-1) = v > 0$ and let α be as above. Then α lands in the Γ-invariants of $E^0[\![x]\!]/\langle p\rangle([p^v](x))$ and the map*

$$L \otimes_{E^0} E^0(BGL_p(\mathbb{F}_q)) \longrightarrow \left(L \otimes_{E^0} E^0(BT_p)^{\Sigma_p}\right) \times \left(L \otimes_{E^0} (E^0[\![x]\!]/\langle p\rangle([p^v](x)))^\Gamma\right)$$

is an isomorphism.

Proof. We will defer an explicit proof that α lands in the Γ-invariants until Chapter 6 although it is implicit in the workings below. Using the generalised character isomorphism we have a diagram

$$\begin{tikzcd}
L \otimes_{E^0} E^0(B(\mathbb{F}_{q^p})^\times) \arrow[r, "\sim"] \arrow[d] & \mathrm{Map}(\mathrm{Rep}(\Theta^*, (\mathbb{F}_{q^p})^\times), L) \arrow[r, equal] \arrow[d] & \mathrm{Map}(\Phi(p^{v+1}), L) \arrow[d] \\
L \otimes_{E^0} E^0(B(\mathbb{F}_q)^\times) \arrow[r, "\sim"] & \mathrm{Map}(\mathrm{Rep}(\Theta^*, (\mathbb{F}_q)^\times), L) \arrow[r, equal] & \mathrm{Map}(\Phi(p^v), L)
\end{tikzcd}$$

But

$$\begin{aligned}
\mathrm{Map}(\Phi(p^{v+1}), L) &= \mathrm{Map}(\Phi(p^v) \textstyle\coprod \Phi(p^{v+1}) \setminus \Phi(p^v), L) \\
&= \mathrm{Map}(\Phi(p^v), L) \times \mathrm{Map}(\Phi(p^{v+1}) \setminus \Phi(p^v), L).
\end{aligned}$$

Thus, using the isomorphism of Lemma 5.28 we have

$$\begin{aligned}
L \otimes_{E^0} \frac{E^0[\![x]\!]}{\langle p\rangle([p^v](x))} &= \ker(L \otimes_{E^0} E^0(B(\mathbb{F}_{q^p})^\times) \to L \otimes_{E^0} E^0(B(\mathbb{F}_q)^\times)) \\
&\simeq \ker(\mathrm{Map}(\Phi(p^{v+1}), L) \to \mathrm{Map}(\Phi(p^v), L)) \\
&= \mathrm{Map}(\Phi(p^{v+1}) \setminus \Phi(p^v), L).
\end{aligned}$$

Finally, Proposition 5.25 gives

$$\begin{tikzcd}
L \otimes_{E^0} E^0(BGL_p(\mathbb{F}_q)) \arrow[r] \arrow[d, "\wr"'] & \left(L \otimes_{E^0} E^0(BT_p)^{\Sigma_p}\right) \times \left(L \otimes_{E^0} (E^0[\![x]\!]/\langle p\rangle([p^v](x)))^\Gamma\right) \arrow[d, "\wr"] \\
\mathrm{Map}((\Phi^p/\Sigma_p)^\Gamma, L) \arrow[r, "\sim"] & \mathrm{Map}((\Phi(p^v)^p/\Sigma_p), L) \times \mathrm{Map}((\Phi(p^{v+1}) \setminus \Phi(p^v))/\Gamma, L)
\end{tikzcd}$$

and the result follows. ☐

In Chapter 6 we will look at the maps of Proposition 5.29 again and see that there is a close relationship between them even before applying the functor $L \otimes_{E^0} -$. However, we will no longer obtain a splitting; the relationship is more subtle.

Chapter 6

The ring $\mathbf{E^0(BGL_d(K))}$

In this chapter we examine the structure of the ring $E^0(BGL_d(K))$ for finite fields K of characteristic different from p. We first consider the low dimensional case where $d < p$ before moving on to study the more complex situations that arise when $d = p$. Our main results will assume that $v_p(|K|^\times) > 0$ so that we have a good understanding of the Sylow p-subgroups of $GL_d(K)$ from Section 3.2.

6.1 Tanabe's calculations

Let l be a prime different to p and $q = l^r$ for some r. Let $\Gamma = \Gamma_q$ be the subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_q)$ generated by $F_q = F^r$, where F is the Frobenius homomorphism $a \mapsto a^l$ of Section 2.1.4. Then Γ acts on $GL_d(\overline{\mathbb{F}}_l)$ component-wise and $GL_d(\mathbb{F}_q) = GL_d(\overline{\mathbb{F}}_l)^\Gamma$. We use the following lemma.

Lemma 6.1. *Let h be any cohomology theory. Let K act on a group G and H be a subgroup of G^K. Then the restriction map $h^*(BG) \to h^*(BH)$ factors through $h^*(BG)_K$, where $h^*(BG)_K$ denotes the coinvariants of the induced action, that is the quotient of $h^*(BG)$ by the ideal $\{a - k^*a \mid a \in h^*(BG), k \in K\}$.*

Proof. Let $k \in K$. Then the commuting diagram

$$\begin{tikzcd} G & H \arrow[l] \arrow[dl] \\ G \arrow[u, "k"] & \end{tikzcd} \quad \text{induces} \quad \begin{tikzcd} h^*(BG) \arrow[r] \arrow[d, "k^*"'] & h^*(BH) \\ h^*(BG) \arrow[ur] & \end{tikzcd}$$

showing that $a - k^*a$ is in the kernel of the restriction map $h^*(BG) \to h^*(BH)$. □

In his paper [Tan95], Tanabe proved the following result.

Proposition 6.2. *$K(n)^*(BGL_d(\mathbb{F}_q))$ is concentrated in even degrees and restriction induces an isomorphism $K(n)^*(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \simeq K(n)^*(BGL_d(\mathbb{F}_q))$.*

Thus we have $K(n)^*(BGL_d(\mathbb{F}_q)) \simeq K(n)^*[\![\sigma_1, \ldots, \sigma_d]\!]/(\sigma_1 - (F_q)^*\sigma_1, \ldots, \sigma_d - (F_q)^*\sigma_d)$. In fact, Tanabe's result lifts to E-theory; that is, we will prove the following.

Proposition 6.3. *$E^*(BGL_d(\mathbb{F}_q))$ is concentrated in even degrees and is free and finitely generated over E^*. Further, the restriction map induces an isomorphism*

$$E^*(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \simeq E^*(BGL_d(\mathbb{F}_q)).$$

To prove this we need a number of intermediate results. Note first that, by Proposition 4.54 and the fact that $K(n)^*(BGL_d(\mathbb{F}_q))$ is concentrated in even degrees, $E^*(BGL_d(\mathbb{F}_q))$ is free over E^* and concentrated in even degrees and, further, $K^*(BGL_d(\mathbb{F}_q)) = K^* \otimes_{E^*} E^*(BGL_d(\mathbb{F}_q))$. Hence it suffices to prove the result in degree 0.

Lemma 6.4. $K^0 \otimes_{E^0} E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma = K^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$.

Proof. Since $E^0(BGL_d(\overline{\mathbb{F}}_l)) = E^0[\![\sigma_1, \ldots, \sigma_d]\!] = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}, \sigma_1, \ldots, \sigma_d]\!]$ (by Corollary 4.82) it follows that $K^0(BGL_d(\overline{\mathbb{F}}_l)) = K^0 \otimes_{E^0} E^0(BGL_d(\overline{\mathbb{F}}_l))$ and we have a diagram

$$\begin{tikzcd}
E^0(BGL_d(\overline{\mathbb{F}}_l)) \arrow[r] \arrow[d, two heads] & E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \arrow[r] \arrow[d, two heads] & E^0(BGL_d(\mathbb{F}_q)) \arrow[d, two heads] \\
K^0(BGL_d(\overline{\mathbb{F}}_l)) \arrow[r] \arrow[dr, two heads] & K^0 \otimes_{E^0} E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \arrow[r] \arrow[d] & K^0(BGL_d(\mathbb{F}_q)) \\
& K^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \arrow[ur, "\sim"'] &
\end{tikzcd}$$

where the second row is reduction modulo $(p, u_1, \ldots, u_{n-1})$. Chasing the diagram we see that the map $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \to K^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ is surjective and, further, it is just reduction modulo the ideal $(p, u_1, \ldots, u_{n-1})$; that is, $K^0 \otimes_{E^0} E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma = K^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$. ☐

Lemma 6.5. *The sequence $p, u_1, \ldots, u_{n-1}$ is regular on $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$.*

Proof. Write σ_i^* for $(F_q)^*\sigma_i$. Then, by [Tan95, Proposition 4.6], $p,\ \sigma_1 - \sigma_1^*, \ldots, \sigma_d - \sigma_d^*$ is a regular sequence on

$$\widehat{K(n)}^0(BGL_d(\overline{\mathbb{F}}_l)) = \mathbb{Z}_p[\![\sigma_1, \ldots, \sigma_d]\!] = \frac{E^0[\![\sigma_1 \ldots, \sigma_d]\!]}{(u_1, \ldots, u_{n-1})} = \frac{E^0(BGL_d(\overline{\mathbb{F}}_l))}{(u_1, \ldots, u_{n-1})}.$$

It follows that $u_1, \ldots, u_{n-1}, p, \sigma_1 - \sigma_1^*, \ldots, \sigma_d - \sigma_d^*$ is regular on $E^0(BGL_d(\overline{\mathbb{F}}_l))$. But, since $E^0(BGL_d(\overline{\mathbb{F}}_l))$ is a Noetherian ring, the corollary to Theorem 16.3 in [Mat89] tells us that $\sigma_1 - \sigma_1^*, \ldots, \sigma_d - \sigma_d^*, p,\ u_1, \ldots, u_{n-1}$ is also regular. Hence $p, u_1, \ldots, u_{n-1}$ is regular on $E^0(BGL_d(\overline{\mathbb{F}}_l))/(\sigma_1 - \sigma_1^*, \ldots, \sigma_d - \sigma_d^*) = E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$. ☐

Lemma 6.6. *$E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ is finitely generated over E^0.*

Proof. Since $E^0(BGL_d(\overline{\mathbb{F}}_l)) = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}, \sigma_1, \ldots, \sigma_d]\!]$ has Krull dimension $d + n$ it follows that $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma/(p, u_1, \ldots, u_{n-1}) = K^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ has Krull dimension 0. Thus, using Lemma 2.15, we see that $K^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ is finite-dimensional over $\mathbb{F}_p$. Thus, an application of Lemma 2.16 shows that $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ is finitely generated over E^0. ☐

Proof of Proposition 6.3. Since $p, u_1, \ldots, u_{n-1}$ is regular on the finitely generated E^0-module $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ an application of Lemma 2.14 shows that $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma$ is free over E^0. Thus the map $E^0(BGL_d(\overline{\mathbb{F}}_l))_\Gamma \to E^0(BGL_d(\mathbb{F}_l))$ is a map of finitely generated free E^0-modules which becomes an isomorphism modulo $(p, u_1, \ldots, u_{n-1})$. Hence, from Proposition 2.12, the map is surjective. But a surjective map of free modules of the same rank must be an isomorphism, and we are done. ☐

6.2 The restriction map $E^0(BGL_d(K)) \to E^0(BT_d)$

Recall that $T_d \simeq (K^\times)^d$ denotes the maximal torus of $GL_d(K)$ and that there is a restriction map $E^0(BGL_d(K)) \to E^0(BT_d)^{\Sigma_d}$. This map plays a large part in our later calculations and we show now that, for all d and K, it is surjective. Put $v = v_p(|K|^\times)$. Then using our complex orientation we have an identification

$$E^0(BT_d) \simeq E^0[\![x_1, \ldots, x_d]\!]/([p^v](x_1), \ldots, [p^v](x_d)).$$

Note that T_d naturally sits inside $\overline{T}_d \simeq (\overline{K}^\times)^d \subseteq GL_d(\overline{K})$. We need a couple of definitions.

Definition 6.7. Let M be a finitely generated free R-module and G a (finite) group acting on M. Then we call M a *permutation module for* G if there is a basis for M over R such that the action of G permutes the basis.

Lemma 6.8. *Let M and G be as above and let S be a basis for M closed under the action of G. For $s \in S$ write*

$$s^G = \sum_{s' \in orb_G(s)} s'.$$

Then the set $\{s^G \mid s \in S\}$ is a basis for M^G.

Proof. Let $m = \sum_{s \in S} m_s s \in M^G$. Then, for $g \in G$, we get $g. \sum_{s \in S} m_s s = \sum_{s \in S} m_s s$ so that $\sum_{s \in S} m_s(g.s) = \sum_{s \in S} m_s s$. Thus, using the fact that G permutes S, we have $m_{g.s} = m_s$ for all s. Hence $m_{s'} = m_s$ for all $s' \in \mathrm{orb}_G(s)$. It follows that m is an R-linear sum of the s^G's. That these are linearly independent follows easily from the fact that S was a basis. $\square$

The basis introduced above for M^G is known as the *basis of orbit sums*, for obvious reasons.

Lemma 6.9. *Let Σ_d act on T_d and $\overline{T}_d$ by permuting the coordinates. Then restriction induces a surjective map $E^0(B\overline{T}_d)^{\Sigma_d} \to E^0(BT_d)^{\Sigma_d}$.*

Proof. Firstly, since the restriction map $E^0(B\overline{T}_d) \to E^0(BT_d)$ is Σ_d-equivariant it induces a map of the Σ_d-invariants. We have identifications $E^0(B\overline{T}_d) \simeq E^0[\![x_1, \ldots, x_d]\!]$ and $E^0(BT_d) \simeq E^0[\![x_1, \ldots, x_d]\!]/([p^v](x_i))$ with restriction being just the obvious quotient map. Using Corollary 4.28, $E^0[\![x_1, \ldots, x_d]\!]/([p^v](x_i))$ has basis $S = \{x_1^{\alpha_1} \ldots x_d^{\alpha_d} \mid 0 \leq \alpha_k < p^{nv}\}$ over E^0 and is a permutation module for Σ_d. Thus we can take the basis of orbit sums for the Σ_d-invariants. It is easy to see that any such basis element can be lifted to a Σ_d-invariant element of $E^0(B\overline{T}_d)$ under our map and we are done. $\square$

Proposition 6.10. *The restriction map $E^0(BGL_d(K)) \to E^0(BT_d)^{\Sigma_d}$ is surjective.*

Proof. Using Corollary 4.82, a system of inclusions induces the diagram

$$\begin{tikzcd} E^0(B\overline{T}_d)^{\Sigma_d} \arrow[d] & E^0(BGL_d(\overline{K})) \arrow[l, "\sim"'] \arrow[d] \\ E^0(T_d)^{\Sigma_d} & E^0(BGL_d(K)) \arrow[l] \end{tikzcd}$$

showing that the composition $E^0(BGL_d(\overline{K})) \to E^0(BT_d)^{\Sigma_d}$ is surjective and hence that $E^0(BGL_d(K)) \to E^0(BT_d)^{\Sigma_d}$ is also surjective. $\square$

6.3 Low dimensions

Here we deal with the case where $d < p$. As usual, we let $q = l^r$ be a power of a prime different to p and let T_d denote the maximal torus of $GL_d(\mathbb{F}_q)$. Our main theorem in this case is as follows.

Theorem B. *If $d < p$ and $v_p(q-1) = v > 0$ then the restriction map $E^0(BGL_d(\mathbb{F}_q)) \to E^0(BT_d)$ induces an isomorphism $E^0(BGL_d(\mathbb{F}_q)) \simeq E^0(BT_d)^{\Sigma_d}$.*

Proof. By Proposition 3.12, T_d is a Sylow p-subgroup of $GL_d(\mathbb{F}_q)$ and it follows that the map $E^0(BGL_d(\mathbb{F}_q)) \to E^0(BT_d)^{\Sigma_d}$ is injective. Further, the image is the whole of $E^0(BT_d)^{\Sigma_d}$ by Proposition 6.10. $\square$

Corollary 6.11. *With the hypotheses of Theorem B we have a presentation*

$$E^0(BGL_d(\mathbb{F}_q)) \simeq \left(\frac{E^0[\![x_1, \ldots, x_d]\!]}{([p^v](x_1), \ldots, [p^v](x_d))} \right)^{\Sigma_d}$$

where $res_{T_d}^{GL_d(K)}(x_i) = euler_l(T_d \xrightarrow{\pi_i} \mathbb{F}_q^\times \rightarrowtail \overline{\mathbb{F}}_l)$ and, as E^0-modules,

$$E^0(BGL_d(\mathbb{F}_q)) \simeq E^0\{\sigma_1^{\alpha_1} \ldots \sigma_d^{\alpha_d} \mid 0 \leq \alpha_1 + \ldots + \alpha_d < p^{nv}\}$$

where σ_i is the i^{th} elementary symmetric function in $x_1, \ldots, x_n$.

Proof. The first statement follows from the corresponding presentation of $E^0(BT_d)^{\Sigma_d}$. The E^0-basis is an application of Proposition 2.9. $\square$

Before we move on to the higher dimensional cases we note that, by the work of Strickland in [Str00], both of $E^0(BGL_d(\mathbb{F}_q))$ and $K^0(BGL_d(\mathbb{F}_q))$ have duality over their respective coefficient rings in the sense of Section 2.1.11. In fact we can verify the latter claim directly.

Lemma 6.12. *Let $N \in \mathbb{N}$ and write $A = \mathbb{F}_p[[x_1, \ldots, x_d]]/(x_1^N, \ldots, x_d^N)$. Then, taking the standard basis for A, the map $A \to \mathbb{F}_p$ given by $\sum_\alpha k_\alpha \mathbf{x}^\alpha \longmapsto k_{N-1,\ldots,N-1}$ is a Frobenius form on A.*

Proof. Note first that

$$\mathrm{soc}(A) = \mathrm{ann}_A(\mathfrak{m}) = \mathrm{ann}_A(x_1, \ldots, x_d) = ((x_1 \ldots x_d)^{N-1}) = \mathbb{F}_p.(x_1 \ldots x_d)^{N-1}.$$

As in the proof of Proposition 2.27, it follows that A has duality over $\mathbb{F}_p$ and that the given map is indeed a Frobenius form on A. $\square$

Corollary 6.13. *For any $N \in \mathbb{N}$, the $\mathbb{F}_p$-algebra $\left(\mathbb{F}_p[[x_1, \ldots, x_d]]/(x_1^N, \ldots, x_d^N)\right)^{\Sigma_d}$ has duality over $\mathbb{F}_p$.*

Proof. Apply Proposition 2.28 with A being the $\mathbb{F}_p$-algebra $\mathbb{F}_p[[x_1, \ldots, x_d]]/(x_1^N, \ldots, x_d^N)$ and $G = \Sigma_d$, noting that A has $\mathbb{F}_p$-basis $\{x_1^{\alpha_1} \ldots x_d^{\alpha_d} \mid 0 \leq \alpha_1, \ldots, \alpha_d < N\}$ and that the map $A \to \mathbb{F}_p$ given by $\sum_\alpha k_\alpha \mathbf{x}^\alpha \longmapsto k_{N-1,\ldots,N-1}$ is a Frobenius form on A. $\square$

Corollary 6.14. *With the hypotheses of Theorem B the algebra $K^0(BGL_d(K))$ has duality over $K^0 = \mathbb{F}_p$.*

Proof. By Proposition 4.55 we have $K^0(BGL_d(K)) = K^0 \otimes_{E^0} E^0(BGL_d(K))$. Since $[p^v](x)$ is a unit multiple of $x^{p^{nv}}$ modulo $(p, u_1, \ldots, u_n)$ we find that

$$K^0(BGL_d(K)) = \mathbb{F}_p[\![x_1, \ldots, x_d]\!]/(x_1^{p^{nv}}, \ldots, x_d^{p^{nv}}).$$

Applying the preceding proposition then gives the result. $\square$

Remark 6.15. The case where $v_p(q-1) = 0$ seems to be quite complicated. However, by Proposition 3.14, we have identified a Sylow p-subgroup P of $GL_d(K)$ and it follows that the restriction map $E^0(BGL_d(K)) \to E^0(BP)$ is injective. It remains to identify the image of the map, although this will be easier said than done.

6.4 Dimension p

As usual, let $q = l^r$ be a power of a prime l different to p and suppose that $v_p(q-1) = v > 0$. We aim to get a handle on $E^0(BGL_p(\mathbb{F}_q))$ by consideration of two comparison maps to more easily understandable rings. As for the low-dimensional case we have the inclusion of the maximal torus $T = T_p$ which induces a map to one of these rings, although unlike for the lower dimensional case this will no longer be an isomorphism. For the other, choosing a basis of $\mathbb{F}_{q^p}^\times$ over $\mathbb{F}_q$ leads to an embedding $\mu : \mathbb{F}_{q^p}^\times \to GL_p(\mathbb{F}_q)$ and we use this to define our second map. It is the interplay between these two maps that will give us the structure theorem in this case.

For the remainder of this section, we will let $\beta : E^0(BGL_p(\mathbb{F}_q)) \to E^0(BT)^{\Sigma_p}$ denote the surjective restriction map of Section 6.2. As in Section 5.2.2, writing $x = \mathrm{euler}_l(\mathbb{F}_{q^p}^\times \rightarrowtail \overline{\mathbb{F}}_l^\times)$ we get $E^0(B\mathbb{F}_{q^p}^\times) \simeq E^0[\![x]\!]/[p^{v+1}](x)$ and hence a quotient map $q : E^0(B\mathbb{F}_{q^p}^\times) \to E^0[\![x]\!]/\langle p\rangle([p^v](x))$ and we denote the target ring here by D. We then let $\alpha = q \circ \mu^* : E^0(BGL_p(\mathbb{F}_q)) \to D$. Note that $\Gamma = \mathrm{Gal}(\mathbb{F}_{q^p}/\mathbb{F}_q)$ acts on $E^0(B\mathbb{F}_{q^p}^\times)$ and hence also on D.

Lemma 6.16. *The map $E^0(BGL_p(\mathbb{F}_q)) \to E^0(B\mathbb{F}_{q^p}^\times)$ lands in the Γ-invariants. Hence the image of α is contained in D^Γ.*

Proof. Let $k \in \mathbb{F}_{q^p}^\times$. Then the $\mathbb{F}_q$-linear isomorphism F_q fits into the diagram

$$\begin{tikzcd} \mathbb{F}_{q^p} \arrow[r, "\times k"] \arrow[d, "F_q"', "\wr"] & \mathbb{F}_{q^p} \arrow[d, "F_q", "\wr"'] \\ \mathbb{F}_{q^p} \arrow[r, "\times k^q"'] & \mathbb{F}_{q^p}. \end{tikzcd}$$

Hence, with our chosen basis for $\mathbb{F}_{q^p}$ over $\mathbb{F}_q$, there is an $F_q \in GL_p(\mathbb{F}_q)$ corresponding to the Frobenius map and the diagram

$$\begin{tikzcd} \mathbb{F}_{q^p}^\times \arrow[r, tail] \arrow[d, "F_q"'] & GL_p(\mathbb{F}_q) \arrow[d, "\mathrm{conj}_{F_q}"] \\ \mathbb{F}_{q^p}^\times \arrow[r, tail] & GL_p(\mathbb{F}_q) \end{tikzcd}$$

commutes. On passing to cohomology, $\mathrm{conj}_{F_q}^*$ is just the identity map so that the map $E^0(BGL_p(\mathbb{F}_q)) \to E^0(B\mathbb{F}_{q^p}^\times)$ lands in the Γ-invariants. The result follows. $\square$

We now state our main theorem in this case.

Theorem C. *Let $q = l^r$ be a power of a prime l different to p and suppose that $v_p(q-1) = v > 0$. Then there are jointly injective E^0-algebra epimorphisms*

$$\begin{tikzcd} & E^0(BGL_p(\mathbb{F}_q)) \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ E^0(BT)^{\Sigma_p} & & D^\Gamma \end{tikzcd}$$

which induce a rational isomorphism

$$\mathbb{Q} \otimes E^0(BGL_p(\mathbb{F}_q)) \xrightarrow{\sim} (\mathbb{Q} \otimes E^0(BT)^{\Sigma_p}) \times (\mathbb{Q} \otimes D^\Gamma).$$

Further, both $\ker(\alpha)$ *and* $\ker(\beta)$ *are E^0-module summands in $E^0(BGL_p(\mathbb{F}_q))$ and the latter is principal. Finally, we have* $\ker(\alpha) = ann(\ker(\beta))$ *and* $\ker(\beta) = ann(\ker(\alpha))$.

The rest of this chapter is devoted to proving this result.

6.4.1 A cyclic p-subgroup of maximal order

Recall that we defined $\alpha : E^0(BGL_p(\mathbb{F}_q)) \to D$ as the composite of the restriction under some embedding $\mathbb{F}_{q^p}^\times \to GL_p(\mathbb{F}_q)$ and a quotient map. For convenience, we choose the embedding a little more carefully to allow us easier analysis of the structure.

Recall, from Section 2.1.4, that we have a fixed embedding $\overline{\mathbb{F}}_l^\times \to S^1$ and this gives us an isomorphism of groups $\mathbb{Z}/p^\infty \xrightarrow{\sim} \{a \in \overline{\mathbb{F}}_l^\times | a^{p^s} = 1 \text{ for some } s\}$. Hence, for each $s \geq 1$, we have canonical generators, a_s say, for the cyclic subgroups C_{p^s} compatible in the sense that $a_{s+1}^p = a_s$.

As in Section 3.3, we let $\gamma = \gamma_p$ denote the standard p-cycle $(1 \dots p) \in \Sigma_p$ and put

$$a = \gamma(a_v, 1, \dots, 1) \in \Sigma_p \wr \mathbb{F}_q^\times \subseteq GL_p(\mathbb{F}_q).$$

We then let $A = \langle a \rangle$ be the subgroup of $GL_p(\mathbb{F}_q)$ generated by a. Note that $a^p = (a_v, \dots, a_v)$ so that $a^{p^{v+1}} = 1$ and A is cyclic of order p^{v+1}.

Now, from Proposition 3.22, A is a maximal abelian p-subgroup of $GL_p(\mathbb{F}_q)$ and any other cyclic subgroup of order p^{v+1} is $GL_p(\mathbb{F}_q)$-conjugate to A. In particular, A is conjugate to the p-part of $\mathbb{F}_{q^p}^\times \simeq C_{p^{v+1}}$. That is, there exists $g \in GL_p(\mathbb{F}_q)$ such that $gAg^{-1} \subseteq \mathbb{F}_{q^p}^\times$. But then, by post composing our chosen embedding $\mathbb{F}_{q^p}^\times \to GL_p(\mathbb{F}_q)$ by $\mathrm{conj}_{g^{-1}}$, we get a new embedding such that A is precisely the p-part of $\mathbb{F}_{q^p}^\times$. Thus we can assume that our original embedding was such that $A \subseteq \mathbb{F}_{q^p}^\times$ and hence we can identify $E^0(B\mathbb{F}_{q^p}^\times)$ with $E^0(BA)$. Thus the map α can be viewed as the composition $GL_p(\mathbb{F}_q) \to E^0(BA) \simeq E^0[\![x]\!]/[p^{v+1}](x) \to D$, where $x = \mathrm{euler}_l(A \rightarrowtail \mathbb{F}_{q^p}^\times \rightarrowtail (\overline{\mathbb{F}}_l)^\times)$.

6.4.2 The ring D^Γ

We defined $D = E^0[\![x]\!]/\langle p \rangle([p^v](x))$ as a quotient ring of $E^0(B\mathbb{F}_{q^p}^\times)$ in the previous section and noted that it inherited an action of $\Gamma = \langle F_q \rangle = \mathrm{Gal}(\mathbb{F}_{q^p}/\mathbb{F}_q) \simeq C_p$. Our first step is to understand how this action works.

Lemma 6.17. *The action of* Γ *on* D *is given by* $F_q.x = [q](x)$.

Proof. Using the embedding $\overline{\mathbb{F}}_l^\times \rightarrowtail S^1$ we get a diagram

$$\begin{tikzcd}
\mathbb{F}_{q^p}^\times \arrow[r] \arrow[d, "F_q"', "\wr"] & S^1 \arrow[d] & z \arrow[d, mapsto] \\
\mathbb{F}_{q^p}^\times \arrow[r] & S^1 & z^q.
\end{tikzcd}$$

On passing to cohomology the result follows. $\square$

To progress, we record a couple of results concerning norm maps. Let $R \hookrightarrow S$ be an extension of rings and suppose that S is finitely generated and free over R of rank n. The *norm map*, $N_{S/R} : S \to R$, is defined by $N_{S/R}(s) = \det(\mu_s)$ where $\mu_s : S \to S$ is multiplication by s.

Lemma 6.18. *Let* $R \hookrightarrow S$ *be an extension of rings and suppose that* S *is finitely generated and free over* R *of rank* n. *Then* $N_{S/R}(s)$ *is divisible by* s *for all* $s \in S$.

Proof. After choosing a basis for S over R let $\chi_s(t) = \det(\mu_s - tI)$ so that $\chi_s(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0$ with $a_i \in R$ for each i. By the Cayley-Hamilton theorem we have $\mu_s^n + a_{n-1}{\mu_s}^{n-1} + \ldots + a_0 = 0$ and hence $s^n + a_{n-1}s^{n-1} + \ldots + a_0 = 0$. Since $a_0 = \chi_s(0) = \det(\mu_s) = N_{S/R}(s)$ we see that $s | N_{S/R}(s)$, as required. $\square$

Recall that any integral domain R has a field of fractions which we denote by $Q(R)$.

Lemma 6.19. *Let* $R \hookrightarrow S$ *be an extension of integral domains and suppose that* S *is finitely generated and free over* R *of rank* n. *Let* $0 \neq s \in S$ *and suppose that* $N_{S/R}(s) = 0$. *Then* $s = 0$.

Proof. Choose a basis for S over R and let $\mu_s : S \to S$ be multiplication by s, so that we can view μ_s as an $n \times n$ matrix over R. Then $\det(\mu_s) = N_{S/R}(s) = 0$ so, by linear algebra over $Q(R)$, there is a non-trivial $u = (u_1, \ldots, u_n) \in Q(R)^n$ with $\mu_s.u = 0$. Write $u_i = v_i/w_i$ for some $v_i, w_i \in R$ with $w_i \neq 0$. Put $w = \prod_i w_i \neq 0$ and $\tilde{u} = wu$. Then $0 \neq \tilde{u} \in R^n$ and $\mu_s.\tilde{u} = w\mu_s.u = 0$. That is, there is a non-zero element $\tilde{u} \in S$ such that $s\tilde{u} = \mu_s(\tilde{u}) = 0$. Hence, since S is an integral domain, $s = 0$. $\square$

Proposition 6.20. *Let* $R \hookrightarrow S$ *be an extension of integral domains and suppose that* S *is finitely generated and free over* R. *Then* $Q(S) = Q(R) \otimes_R S$.

Proof. Let $\phi : Q(R) \otimes_R S \to Q(S)$ be the ring map $\phi(k \otimes s) = ks$. We show that ϕ is an isomorphism. For injectivity, take $a = \sum_i (b_i \otimes c_i/d_i) \in \ker(\phi)$ with $b_i \in S$ and $c_i, d_i \in R$ with $d_i \neq 0$. Put $d = \prod_i d_i$ (necessarily non-zero) and $\bar{d}_i = \prod_{j \neq i} d_j$. Then we have $a = \sum_i (b_i \otimes c_i\bar{d}_i/d) = \sum_i (b_i c_i \bar{d}_i \otimes 1/d)$ and so, since $\phi(a) = 0$, we get $(\sum_i b_i c_i \bar{d}_i)/d = 0$ in $Q(S)$. Thus, we see $\sum_i b_i c_i \bar{d}_i = 0$ and $a = (\sum_i b_i c_i \bar{d}_i) \otimes 1/d = 0$, as required.

To show that ϕ is surjective, take $a/b \in Q(S)$ and let $c = N_{S/R}(b) \in R$. Since $b \neq 0$ we see from Lemma 6.19 that $c \neq 0$ so $1/c$ exists in $Q(R)$. By Lemma 6.18 we have $c = b.\bar{b}$ for some $\bar{b} \in S$ and then $a/b = a\bar{b}/c = \phi(1/c \otimes a\bar{b})$. Hence ϕ is an isomorphism, as claimed. $\square$

We now have the tools we need to study the rings D and D^Γ and, in particular, their module structures over E^0.

Definition 6.21. We will let $N = (p^{n(v+1)} - p^{nv})/p$ and define $y = \prod_{k=0}^{p-1}[1 + kp^v](x) \in D$. We also let $g(x) = g_{v+1}(x)/g_v(x)$ be the Weierstrass polynomial of degree Np which is a unit multiple of $\langle p\rangle([p^v](x))$ in $E^0[\![x]\!]$. Note that $D = E^0[\![x]\!]/g(x)$.

Lemma 6.22. *With the notation above, $g(x)$ is monic and irreducible over E^0.*

Proof. By definition $g(x)$ is monic. Note that $g(x) \sim [p^{v+1}](x)/[p^v](x) = \langle p\rangle([p^v](x))$ so that $g(0) \sim p$, say $g(0) = ap$ for some unit $a \in (E^0)^\times$. Then, since $g(x) = x^{Np}$ modulo $(p, u_1, \ldots, u_{n-1})$, an application of Eisenstein's criterion ([Mat89, p228]) shows that $g(x)$ is irreducible over E^0. □

Corollary 6.23. *D is an integral domain and is free of rank Np over E^0.*

Proof. The first result is immediate since $D = E^0[\![x]\!]/g(x)$ and the second is an application of the Weierstrass preparation theorem. □

Lemma 6.24. *D is free over E^0 with basis $S = \{x^i y^j \mid 0 \leq i < p,\ 0 \leq j < N\}$.*

Proof. We first show that S generates D over E^0. Working modulo $(p, u_1, \ldots, u_{n-1})$ we have $g_{v+1}(x) = x^{n(v+1)}$ and $g_v(x) = x^{nv}$ so that $g(x) = x^{n(v+1)-nv} = x^{Np}$. Hence $D/(p, u_1, \ldots, u_{n-1}) = \mathbb{F}_p[\![x]\!]/x^{Np} = \mathbb{F}_p\{1, x, \ldots, x^{Np-1}\}$ as an $\mathbb{F}_p$-vector space. Now, since $y = \prod_{k=0}^{p-1}[1 + kp^v](x) = \prod_{k=0}^{p-1}((1 + kp^v)x + O(2))$ we find that, mod $(p, u_1, \ldots, u_{n-1})$, we have $y = x^p + O(p+1)$. It follows easily that $\{x^i y^j \mid 0 \leq i < p, 0 \leq j < N\}$ is also a basis for $D/(p, u_1, \ldots, u_{n-1})$. Applying Lemma 2.16 gives us a generating set S for D over E^0. But D is free over E^0 of rank $Np = |S|$ so that S is a basis for D. □

Lemma 6.25. *Let $y' = \prod_{\gamma \in \Gamma} \gamma.x \in D^\Gamma$. Then $y = y'$ and, in particular, y is Γ-invariant.*

Proof. Since $\Gamma = \langle F_q \rangle$ is cyclic of order p we find that $y' = \prod_{k=0}^{p-1}[q^k](x)$. But, by assumption, $q = 1 + ap^v$ for some a not divisible by p and it follows that q generates the subgroup $1 + p^v\mathbb{Z}/p^{v+1} \subseteq (\mathbb{Z}/p^{v+1})^\times$. Thus $y' = \prod_{k=0}^{p-1}[1 + kp^v](x) = y$. □

We now aim to understand D^Γ. We already know that $y \in D^\Gamma$.

Lemma 6.26. *Γ acts on $Q(D)$ by $\gamma.\dfrac{a}{b} = \dfrac{\gamma.a}{\gamma.b}$ and $Q(D)^\Gamma = Q(D^\Gamma)$.*

Proof. It is clear that we have an inclusion $Q(D^\Gamma) \hookrightarrow Q(D)$ which lands in the Γ-invariants. It remains to show that the map is surjective. Using Proposition 6.20 we have $Q(D) = Q(D^\Gamma) \otimes_{D^\Gamma} D$. Take any $\frac{ac}{b} = \frac{a}{b} \otimes c \in Q(D)^\Gamma \otimes_{D^\Gamma} D$, where $a, b \in D^\Gamma$ and $c \in D$. Then, for all $\gamma \in \Gamma$ we have $\gamma.\frac{ac}{b} = \frac{ac}{b}$ so that $\frac{a\gamma.c}{b} = \frac{ac}{b}$ whereby $\gamma.c = c$. Thus $c \in D^\Gamma$ and $\frac{ac}{b} \in Q(D^\Gamma)$. □

Lemma 6.27. *$Q(D)$ has dimension p over $Q(D^\Gamma)$.*

Proof. We can embed $\Gamma \rightarrowtail \mathrm{Gal}(Q(D)/Q(E^0))$ and so, by Galois theory, $Q(D)$ has dimension $|\Gamma| = p$ over $Q(D)^\Gamma = Q(D^\Gamma)$. □

Proposition 6.28. *D is free over D^Γ with basis $\{1, x, \ldots, x^{p-1}\}$.*

Proof. By Lemma 6.24 we know that $S = \{x^i y^j \mid 0 \leq i < p,\ 0 \leq j < N\}$ generates D over E^0. Since $y \in D^\Gamma$ we find that the map $D^\Gamma\{1, x, \ldots, x^{p-1}\} \to D$ is surjective. Applying the functor $Q(D^\Gamma) \otimes_{D^\Gamma} -$ we get a map $Q(D^\Gamma)\{1, x, \ldots, x^{p-1}\} \to Q(D)$ which, by right exactness, is also surjective. But, by Lemma 6.27, both source and target are $Q(D^\Gamma)$-vector spaces of dimension p and therefore the map is an isomorphism. Consider the diagram

$$\begin{tikzcd} D^\Gamma\{1, x, \ldots, x^{p-1}\} \arrow[r, two heads] \arrow[d] & D \arrow[d] \\ Q(D^\Gamma)\{1, x, \ldots, x^{p-1}\} \arrow[r, "\sim"] & Q(D). \end{tikzcd}$$

Since D^Γ is an integral domain, the left-hand map is injective. Thus we see that the map $D^\Gamma\{1, x, \ldots, x^{p-1}\} \to D$ is also injective and hence is an isomorphism. □

Proposition 6.29. *D^Γ is free over E^0 with basis $\{1, y, \ldots, y^{N-1}\}$ and there is a polynomial $h(t) \in E^0[t]$ of degree N such that $D^\Gamma = E^0[\![y]\!]/h(y)$.*

Proof. For the first claim, note that we have a map $E^0\{1, y, \ldots, y^{N-1}\} \to D^\Gamma$ and the diagram

$$\begin{tikzcd} (E^0\{1, y, \ldots, y^{N-1}\})\{1, x, \ldots, x^{p-1}\} \arrow[r] \arrow[d, "\wr"'] & D^\Gamma\{1, x, \ldots, x^{p-1}\} \arrow[d, "\wr"] \\ E^0\{S\} \arrow[r, "\sim"] & D \end{tikzcd}$$

shows that it must be an isomorphism. For the second claim, write $y^N = \sum_i a_i y^i$ for unique $a_i \in E^0$; then the polynomial $h(y) = y^N - \sum_i a_i y^i$ does the job. □

Lemma 6.30. *$D^\Gamma/(y, u_1, \ldots, u_{n-1})$ is a one dimensional vector space over $\mathbb{F}_p$.*

Proof. The isomorphism of D^Γ-modules $D^\Gamma\{1, x, \ldots, x^{p-1}\} \xrightarrow{\sim} D$ induces a module isomorphism $D^\Gamma/(y, u_1, \ldots, u_{n-1})\{1, x, \ldots, x^{p-1}\} \xrightarrow{\sim} D/(y, u_1, \ldots, u_{n-1})$. But q is coprime to p and so $y = \prod_{k=0}^{p-1}[q^k](x)$ is a unit multiple of $\prod_{k=0}^{p-1} x = x^p$ in D. Hence $D/(y, u_1, \ldots, u_{n-1}) = D/(x^p, u_1, \ldots, u_{n-1})$. It then follows that

$$x^{p-1}D/(y, u_1, \ldots, u_{n-1}) = x^{p-1}D/(x^p, u_1, \ldots, u_{n-1}) \simeq D^\Gamma/(y, u_1, \ldots, u_{n-1})$$

so that $D^\Gamma/(y, u_1, \ldots, u_{n-1})\{1, x, \ldots, x^{p-2}\} \simeq D/(x^{p-1}, u_1, \ldots, u_{n-1})$. We can continue in this way to see that $D^\Gamma \simeq D/(x, u_1, \ldots, u_{n-1})$. But $g(x) = p \bmod x$ whereby

$$D^\Gamma/(y, u_1, \ldots, u_{n-1}) \simeq D/(x, u_1, \ldots, u_{n-1}) = E^0[\![x]\!]/(x, p, u_1, \ldots, u_{n-1}) \simeq \mathbb{F}_p.$$ □

Proposition 6.31. *D^Γ is a regular local ring and $y, u_1, \ldots, u_{n-1}$ a system of parameters.*

Proof. As D^Γ is finitely generated over E^0 it follows that the Krull dimension of D^Γ is equal to the Krull dimension of E^0, namely n. Thus, since the maximal ideal of D^Γ is generated by $y, u_1, \ldots, u_{n-1}$ it follows that $\mathrm{embdim}(D^\Gamma) \leq n$ and hence that D^Γ is a regular local ring. □

Proposition 6.32. *The map $\alpha : E^0(BGL_p(\mathbb{F}_q)) \to D^\Gamma$ sends σ_i to the i^{th} elementary symmetric function in $x, [q](x), \ldots, [q^{p-1}](x)$. Further, α is surjective.*

Proof. Recall that an application of Lemma 5.17 gives us the isomorphism of $\overline{\mathbb{F}}_l$-vector spaces $\psi : \overline{\mathbb{F}}_l \otimes_{\mathbb{F}_q} \mathbb{F}_{q^p} \xrightarrow{\sim} \overline{\mathbb{F}}_l^p$, $a \otimes b \mapsto (ab, ab^q, \ldots, ab^{q^{p-1}})$. Thus there is $g = g_\psi \in GL_p(\overline{\mathbb{F}}_l)$ such that

$$\begin{tikzcd}
\mathbb{F}_{q^p}^\times \arrow[r] \arrow[d] & GL_p(\overline{\mathbb{F}}_l) \arrow[d, "\mathrm{conj}_g"] \\
(\overline{\mathbb{F}}_l^\times)^p \arrow[r] & GL_p(\overline{\mathbb{F}}_l)
\end{tikzcd}$$

commutes, where the left hand map is $a \mapsto (a, a^q, \ldots, a^{q^{p-1}})$. Passing to cohomology gives a commutative diagram

$$\begin{tikzcd}
E^0(BGL_p(\overline{\mathbb{F}}_l)) \arrow[r] \arrow[dr] & E^0(B\mathbb{F}_{q^p}^\times) \arrow[r] & D^\Gamma \\
& E^0(B(\overline{\mathbb{F}}_l^\times)^p) \arrow[u] & E^0[\![x_1, \ldots, x_p]\!] \arrow[l, "\sim"']
\end{tikzcd}$$

where the map $E^0(B(\overline{\mathbb{F}}_l^\times)^p) \to E^0(B\mathbb{F}_{q^p}^\times)$ sends x_i to $[q^{i-1}](x)$. Remembering that

$$E^0(BGL_p(\overline{\mathbb{F}}_l)) \simeq E^0(BT)^{\Sigma_p} = E^0[\![\sigma_1, \ldots, \sigma_p]\!]$$

we see that $\alpha(\sigma_i)$ is the i^{th} elementary symmetric function in $x, [q](x), \ldots, [q^{p-1}](x)$; in particular, $\alpha(\sigma_p) = y$. Since y generates D^Γ, we are done. ☐

6.4.3 An important subgroup

Recall that, from Section 3.2, there is a subgroup $N = N_p = \Sigma_p \wr \mathbb{F}_q^\times$ of $GL_p(\mathbb{F}_q)$. The cohomology of wreath products is fairly well understood (see [Nak61]); we will use methods similar to those of [Hun90] to calculate $E^0(BN)$. The structure of such rings is usually expressed in terms of standard euler classes, but we get analogous results with our l-euler classes. We begin with a couple of standard results from group cohomology.

Lemma 6.33 (Shapiro's lemma)**.** *Let R be a ring and H be a subgroup of G. Then $H^*(G; R[G/H]) = H^*(H; R)$, where R is a trivial H-module.*

Proof. This is [Wei94, 6.3.2] with $A = R$. ☐

Lemma 6.34. *Let G be a finite group and M a G-module. Then $|G|.H^i(G; M) = 0$ for all $i > 0$. In particular, if multiplication by $|G|$ is an isomorphism $M \to M$ then $H^i(G; M) = 0$.*

Proof. This is [Wei94, Theorem 6.5.8]. ☐

Lemma 6.35. *Let S be a set and let Σ_p act on S^p in the usual way. If $s \in S^p$ then either $s \in (S^p)^{\Sigma_p} = \Delta(S)$ or p divides $|orb_{\Sigma_p}(s)|$.*

Proof. Let H be a subgroup of Σ_p and suppose that s is fixed by H. If p divides the order of H then H contains a cyclic subgroup of order p necessarily generated by a p-cycle, σ say. But σ acts transitively on S^p so that we must have $s = (s_1, \ldots, s_1) \in \Delta(S)$. Thus, for $s \in S^p$, either $s \in \Delta(S)$ or $\mathrm{stab}_{\Sigma_p}(s) = H$ for some subgroup $H \subseteq \Sigma_p$ with order not divisible by p. The result follows. ☐

Lemma 6.36. *We have $H^*(\Sigma_p; \mathbb{Z}_p) \simeq \mathbb{Z}_p[\![z]\!]/pz$ for a class z in degree $2p - 2$.*

Proof. As in Proposition 4.83, $H^*(B\Sigma_p;\mathbb{Z}_p) \simeq H^*(BC_p;\mathbb{Z}_p)^{\mathrm{Aut}(C_p)}$, where $H^*(BC_p;\mathbb{Z}_p) \simeq \mathbb{Z}_p[\![x]\!]/px$ with x in degree 2 and $\mathrm{Aut}(C_p) \simeq (\mathbb{Z}/p)^\times$ acting by $k.x = kx$. Thus we have $H^*(B\Sigma_p;\mathbb{Z}_p) = (\mathbb{Z}_p[\![x]\!]/px)^{\mathrm{Aut}(C_p)} = \mathbb{Z}_p[\![z]\!]/pz$ where $z = \prod_{k\in(\mathbb{Z}/p)^\times}(kx) = -x^{p-1}$. Using general theory (see, for example, [Wei94]) we identify $H^*(B\Sigma_p;\mathbb{Z}_p)$ with $H^*(\Sigma_p;\mathbb{Z}_p)$ and the result follows. ☐

Lemma 6.37. *There are sets B' and T such that*

$$H^*(B\Sigma_p; E^0(B(\mathbb{F}_q^\times)^p)) \simeq E^0\{B'\}^{\Sigma_p} \oplus (E^0[\![z]\!]/pz)\{T\},$$

where z is in degree $2p-2$.

Proof. As before, we can identify the ring $H^*(B\Sigma_p; E^0(B(\mathbb{F}_q^\times)^p))$ with the group cohomology $H^*(\Sigma_p; E^0(B(\mathbb{F}_q^\times)^p))$. We let $B = \{x_1^{\alpha_1}\dots x_p^{\alpha_p} \mid 0 \le \alpha_1,\dots,\alpha_p < p^{nv}\} \subseteq E^0(B(\mathbb{F}_q^\times)^p)$ and note that $E^0(B(\mathbb{F}_q^\times)^p) = E^0\{B\}$ so that we can apply Lemma 6.35 to get $B = T\cup B'$, where $T = B^{\Sigma_p} = \{x_1^\alpha\dots x_p^\alpha \mid 0\le\alpha<p^{nv}\}$ and B' is a disjoint union of orbits of size divisible by p.

Now, each orbit in B' is of the form Σ_p/H for some H with order not divisible by p, and $H^*(\Sigma_p; E^0[\Sigma_p/H]) \simeq H^*(H;E^0)$ by Lemma 6.33. But, since $|H|$ is invertible in E^0, we find that $H^i(\Sigma_p;E^0[\Sigma_p/H]) = 0$ for all $i>0$. Hence $H^i(\Sigma_p;E^0\{B'\}) = 0$ for all $i>0$. Further, $H^0(\Sigma_p;E^0\{B'\}) = E^0\{B'\}^{\Sigma_p}$.

We now see that $H^*(\Sigma_p;E^0\{B\}) = H^*(\Sigma_p;E^0\{B'\}) \oplus H^*(\Sigma_p;E^0\{T\})$, where the latter summand is just $H^*(\Sigma_p;\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} E^0\{T\}$; that is,

$$H^*(B\Sigma_p; E^0(B(\mathbb{F}_q^\times)^p) \simeq E^0\{B'\}^{\Sigma_p} \oplus (E^0[\![z]\!]/pz)\{T\},$$

where z is in degree $2p-2$. ☐

We are now in a position to establish the cohomology of $N = \Sigma_p \wr \mathbb{F}_q^\times$. Note first that the projection $N \to \Sigma_p$ makes $E^*(BN)$ into a $E^*(B\Sigma_p)$-module. Recall, from Section 4.3.6, that the embedding $C_p \rightarrowtail S^1$ gives a class $w \in E^0(BC_p)$ such that $E^0(BC_p) = E^0[\![w]\!]/[p](w)$ and $E^0(B\Sigma_p) \simeq E^0(BC_p)^{\mathrm{Aut}(C_p)} \simeq E^0[\![d]\!]/df(d)$ where $d = -w^{p-1}$ and $f(d) = \langle p\rangle(w)$.

Lemma 6.38. *In $E^0(B\Sigma_p) = E^0[\![d]\!]/df(d)$ we have $pd \in (d^2)$.*

Proof. Since $df(d) = 0$ and $f(0) = p$, we find $pd = f(0)d = f(d)d - f(0)d = (f(d)-f(0))d$ which is divisible by d^2, as claimed. ☐

Proposition 6.39. *Let $J = \{\alpha\in\mathbb{N}^p \mid 0\le\alpha_1\le\ldots\le\alpha_p<p^{nv} \text{ and } \alpha_1<\alpha_p\}$. Then there is an isomorphism of $E^0(B\Sigma_p)$-modules*

$$E^0(BN) \simeq E^0(B\Sigma_p)\{c_p^i \mid 0\le i<p^{nv}\} \oplus (E^0(B\Sigma_p)/d)\{b_\alpha \mid \alpha\in J\}$$

where $c_p = euler_l(N \hookrightarrow GL_p(\mathbb{F}_q) \hookrightarrow GL_p(\overline{\mathbb{F}}_l))$ and $b_\alpha = tr^N_{(\mathbb{F}_q^\times)^p}(x_1^{\alpha_1}\dots x_p^{\alpha_p})$.

Proof. First note that in $E^0(BN)$,

$$d.b_\alpha = d.\mathrm{tr}^N_{(\mathbb{F}_q^\times)^p}(x_1^{\alpha_1}\dots x_p^{\alpha_p}) = \mathrm{tr}^N_{(\mathbb{F}_q^\times)^p}(\mathrm{res}^N_{(\mathbb{F}_q^\times)^p}(d)x_1^{\alpha_1}\dots x_p^{\alpha_p}) = 0$$

since the composite $(\mathbb{F}_q^\times)^p \to N \to \Sigma_p$ is zero. Thus, writing

$$R = E^0(B\Sigma_p)\{c_p^i \mid 0\le i<p^{nv}\} \oplus (E^0(B\Sigma_p)/d)\{b_\alpha \mid \alpha\in J\}$$

there is an evident (well defined) map of $E^0(B\Sigma_p)$-modules $\phi : R \to E^0(BN)$. We introduce a filtration on R by

$$\begin{aligned} F^0 R &= R, \\ F^1 R = \ldots = F^{2p-2} R &= Rd, \\ F^{2p-1} R = \ldots = F^{4p-4} R &= Rd^2, \ \ldots \quad . \end{aligned}$$

Then

$$\frac{F^{k(2p-2)}R}{F^{(k+1)(2p-2)}R} \quad \simeq \quad (Rd^k/Rd^{k+1}) = \begin{cases} E^0\{c_p^i \mid 0 \le i < p^{nv}\} \oplus E^0\{b_\alpha \mid \alpha \in J\} & \text{for } k = 0 \\ (E^0/p)\{c_p^i \mid 0 \le i < p^{nv}\}d^k & \text{for } k > 0. \end{cases}$$

We use the spectral sequence $H^*(B\Sigma_p; E^*(B(\mathbb{F}_q^\times)^p)) \Rightarrow E^*(BN)$ associated to the fibration $B(\mathbb{F}_q^\times)^p \to BN \to B\Sigma_p$. By Lemma 6.37, $H^*(\Sigma_p; E^*(B(\mathbb{F}_q^\times)^p))$ is in even degrees and the spectral sequence collapses. Thus, we have a filtration $E^0(BN) = F_0 \geqslant F_{2p-2} \geqslant F_{4p-4} \geqslant \ldots$ with $F_{k(2p-2)}/F_{(k+1)(2p-2)} = E_\infty^{k(2p-2),0}$; that is,

$$F_{k(2p-2)}/F_{(k+1)(2p-2)} = \begin{cases} E^0\{T\} \oplus E^0\{B'\}^{\Sigma_p} & \text{for } k = 0 \\ (E^0/p)\{T\}z^k & \text{for } k > 0. \end{cases}$$

It remains to show that ϕ induces an isomorphism

$$\frac{F^{k(2p-2)}R}{F^{(k+1)(2p-2)}R} \simeq \frac{F_{k(2p-2)}}{F_{(k+1)(2p-2)}}$$

for all k since, if so, an application of the five-lemma ([Hat02, p129]) gives an isomorphism

$$\frac{F^0 R}{F^{k(2p-2)}R} \simeq \frac{F_0}{F_{k(2p-2)}}$$

and, on taking limits, an isomorphism $R = F^0 R \simeq F_0 = E^0(BN)$.

Firstly note that the map $E^0(BN) \to F_0/F_{2p-2} = E^0\{T\} \oplus E^0\{B'\}^{\Sigma_p} \subseteq E^0(B(\mathbb{F}_q^\times)^p)$ is just the restriction map (see, for example, [McC01]). An application of the double coset formula shows that $\mathrm{res}^N_{(\mathbb{F}_q^\times)^p}(b_\alpha) = \sum_{\sigma \in \Sigma_p} \sigma.x_1^{\alpha_1} \ldots x_p^{\alpha_p}$ so that the images of b_α ($\alpha \in J$) are precisely the basis elements of $E^0\{B'\}^{\Sigma_p}$, where the latter is given the basis of orbit sums. Further, the restriction of the l-euler class c_p to $E^0(B(\mathbb{F}_q^\times)^p)$ is just $x_1 \ldots x_p$, so that the classes c_p^i ($0 \le i < p^{nv}$) give precisely the set T. Similarly, the class $d^k c_p^j \in F_{k(2p-2)}$ lifts $(x_1 \ldots x_p)^j z^k \in F_{k(2p-2)}/F_{(k+1)(2p-2)}$. The result follows. □

6.4.4 Summary of notation

We summarise the notation for the generators that will be used to study $E^0(BGL_p(\mathbb{F}_q))$.

- We have $v = v_p(q-1)$ which is assumed to be positive.
- We let T denote the maximal torus of $GL_p(\mathbb{F}_q)$ and have

  $$E^0(BT) = E^0[\![x_1 \ldots, x_p]\!]/([p^v](x_1), \ldots, [p^v](x_p)).$$

  We write β for the surjective restriction map $E^0(BGL_p(\mathbb{F}_q)) \to E^0(BT)^{\Sigma_p}$.
- We let Δ denote the diagonal subgroup of T and write $E^0(B\Delta) = E^0[\![x]\!]/[p^v](x)$, where $x = \mathrm{euler}_l(\Delta \simeq \mathbb{F}_q^\times \rightarrowtail (\overline{\mathbb{F}}_l)^\times)$. We also let Δ_p denote the p-part of Δ.

- We let A be the maximal cyclic p-subgroup of $GL_p(\mathbb{F}_q)$ of Section 6.4.1 and, remembering that we can view A as a subgroup of $\mathbb{F}_q^\times$, we have $E^0(BA) \simeq E^0[\![x]\!]/[p^{v+1}](x)$ where $x = \mathrm{euler}_l(A \rightarrowtail \mathbb{F}_{q^p}^\times \rightarrowtail (\overline{\mathbb{F}}_l)^\times)$. Note that this notation is consistent with that of $E^0(B\Delta) = E^0(B\Delta_p)$ since Δ_p sits inside A in a compatible way. We write D for the quotient ring $E^0(BA)/\langle p\rangle([p^v](x))$ and α for the surjective map $E^0(BGL_p(\mathbb{F}_q)) \to D^\Gamma$.
- We let C_p denote the cyclic subgroup of order p. Using the embedding $(\overline{\mathbb{F}}_l)^\times \rightarrowtail S^1$ we get an embedding $C_p \rightarrowtail (\overline{\mathbb{F}}_l)^\times$ and we write $E^0(BC_p) = E^0[\![w]\!]/[p](w)$, where $w = \mathrm{euler}_l(C_p \rightarrowtail (\overline{\mathbb{F}}_l)^\times)$.
- We write $E^0(B\Sigma_p) = E^0[\![d]\!]/df(d)$ as in Proposition 4.88, where the restriction of d to $E^0(BC_p)$ is $-w^{p-1}$ and $f(d)$ restricts to $\langle p\rangle(w)$.
- As in the previous section, we write $N = \Sigma_p \wr \mathbb{F}_q^\times$ and have
$$E^0(BN) \simeq E^0(B\Sigma_p)\{c_p^i \mid 0 \le i < p^{nv}\} \oplus E^0\{b_\alpha \mid \alpha \in J\}$$
where $c_p = \mathrm{euler}_l(N \hookrightarrow GL_p(\mathbb{F}_q) \hookrightarrow GL_p(\overline{\mathbb{F}}_l))$, $b_\alpha = \mathrm{tr}^{N}_{(\mathbb{F}_q^\times)^p}(x_1^{\alpha_1} \dots x_p^{\alpha_p})$ and
$$J = \{\alpha \in \mathbb{N}^p \mid 0 \le \alpha_1 \le \dots \le \alpha_p < p^{nv} \text{ and } \alpha_1 < \alpha_p\}\,.$$

Recall the results of Section 3.3, namely that every p-subgroup of N is subconjugate to one of A, $\Sigma_p \times \Delta$ or T. We begin with the diagram of inclusions below.

$$\begin{tikzcd}
 & & GL_p(\mathbb{F}_q) & & \\
 & & N \arrow[u] & & \\
T \arrow[uurr, bend left] \arrow[urr] & & \Sigma_p \times \Delta \arrow[u] & & A \arrow[uull, bend right] \arrow[ull] \\
 & & \Delta_p \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Applying the functor $E^0(B-)$ we get the following diagram.

$$\begin{tikzcd}
 & E^0(BGL_p(\mathbb{F}_q)) \arrow[d] \arrow[dl, bend right, "\beta"'] \arrow[dr, bend left] & \\
 & E^0(BN) \arrow[dl, "\psi_1"'] \arrow[d, "\psi_2"] \arrow[dr, "\psi_3"] & \\
E^0(BT) \arrow[d, equal] & E^0(B\Sigma_p) \otimes_{E^0} E^0(B\Delta) \arrow[d, equal] & E^0(BA) \arrow[d, equal] \\
\dfrac{E^0[\![x_1 \dots x_p]\!]}{([p^v](x_1), \dots, [p^v](x_p))} \arrow[dr] & \dfrac{E^0[\![d, x]\!]}{(df(d), [p^v](x))} \arrow[d] & \dfrac{E^0[\![x]\!]}{[p^{v+1}](x)} \arrow[dl] \\
 & E^0(B\Delta_p) = \dfrac{E^0[\![x]\!]}{[p^v](x)} &
\end{tikzcd}$$

Proposition 6.40. *The maps ψ_1, ψ_2 and ψ_3 shown above are jointly injective.*

Proof. This follows from Section 3.3 and Corollary 4.72: any abelian p-subgroup of N is subconjugate to one of T, A and $\Sigma_p \times \Delta$. $\square$

Hence we should be able to get a good understanding of the multiplicative structure of $E^0(BN)$ by studying the maps ψ_1, ψ_2 and ψ_3. As well as looking at the usual generators of $E^0(BN)$, we will be particularly interested in a class t defined below.

Proposition 6.41. *There is a unique class $t \in E^0(BGL_p(\overline{\mathbb{F}}_l))$ which restricts to $\prod_i [p^v](x_i)$ in $E^0(B(\overline{\mathbb{F}}_l^\times)^p) \simeq E^0[\![x_1, \ldots, x_p]\!]$. Further, writing t for the restriction of this class to $E^0(BGL_p(\mathbb{F}_q))$, we find that $t \in \ker(\beta)$.*

Proof. By the results of Tanabe we have $E^0(BGL_p(\overline{\mathbb{F}}_l)) = E^0(B(\overline{\mathbb{F}}_l^\times)^p)^{\Sigma_p}$, where Σ_p acts by permuting the x_i, and it is clear that $\prod_i [p^v](x_i)$ is Σ_p-invariant. For the second claim, the commutative diagram

$$\begin{array}{ccccc} E^0(BGL_p(\overline{\mathbb{F}}_l)) & \longrightarrow & E^0(B(\overline{\mathbb{F}}_l^\times)^p) & = & E^0[\![x_1, \ldots, x_p]\!] \\ \downarrow & & \downarrow & & \downarrow \\ E^0(BGL_p(\mathbb{F}_q)) & \longrightarrow & E^0(BT) & = & \dfrac{E^0[\![x_1, \ldots, x_p]\!]}{([p^v](x_1), \ldots, [p^v](x_p))} \end{array}$$

shows that $\beta(t) = \prod_i [p^v](x_i) = 0$ in $E^0(BT)$. $\square$

For the remainder of this chapter we will write I for the ideal of $E^0(BGL_p(\mathbb{F}_q))$ generated by t. Then, by the previous result, we have $I \subseteq \ker(\beta)$. Later we will find that $I = \ker(\beta)$; that is, $\ker(\beta)$ is a principal ideal generated by t.

The following proposition shows the images of the key elements of $E^0(BN)$ under each of the maps ψ_1, ψ_2 and ψ_3 and will be proved in the subsequent section.

Proposition 6.42. *With the notation above, the following table shows the images of the classes d, c_p, b_α ($\alpha \in J$) and t of $E^0(BN)$ under the maps ψ_1, ψ_2 and ψ_3.*

Map / target	d	c_p	b_α ($\alpha \in J$)	t
ψ_1 / $E^0(BT)$	0	$x_1 \ldots x_p$	$\sum_{\sigma \in \Sigma_p} \sigma.(x_1^{\alpha_1} \ldots x_p^{\alpha_p})$	0
ψ_2 / $E^0(B(\Sigma_p \times \Delta))$	d	$\prod_{k=0}^{p-1} (x +_F [k](w))$	$(p-1)! f(d) x^{\sum \alpha_i}$	0
ψ_3 / $E^0(BA)$	$-[p^v](x)^{p-1}$	$\prod_{k=0}^{p-1} [1 + kp^v](x)$	$(p-1)! x^{\sum \alpha_i} \langle p \rangle([p^v](x))$	$[p^v](x)^p$

6.4.5 Tracking the key classes in $E^0(BN)$

Proposition 6.43. *In $E^0(BT) = E^0[\![x_1, \dots, x_p]\!]/([p^v](x_i))$ we have*

$$\psi_1(d) = 0,\ \psi_1(c_p) = x_1 \dots x_p \text{ and } \psi_1(b_\alpha) = \sum_{\sigma \in \Sigma_p} \sigma.(x_1^{\alpha_1} \dots x_p^{\alpha_p}).$$

Proof. For the first statement, the composition $T \rightarrowtail N \twoheadrightarrow \Sigma_p$ is trivial and hence, since d is the restriction of a class in $E^0(B\Sigma_p)$, it follows that $\psi_1(d) = 0$. For the second statement we have $\psi_1(c_p) = \psi_1(\mathrm{euler}_l(N \hookrightarrow GL_p(\overline{\mathbb{F}}_l)) = \mathrm{euler}_l(T \hookrightarrow GL_p(\overline{\mathbb{F}}_l)) = x_1 \dots x_p$. For the final statement, using the double-coset formula (Lemma 4.61) we get

$$\begin{aligned} \psi_1(b_\alpha) &= \mathrm{res}_T^N \mathrm{tr}_T^N (x_1^{\alpha_1} \dots x_p^{\alpha_p}) \\ &= \sum_{\sigma \in T \backslash N / T} \mathrm{tr}_{T \cap T}^T \mathrm{res}_{T \cap T}^T (\mathrm{conj}_\sigma^*(x_1^{\alpha_1} \dots x_p^{\alpha_p})) \\ &= \sum_{\sigma \in \Sigma_p} \sigma.(x_1^{\alpha_1} \dots x_p^{\alpha_p}), \end{aligned}$$

as claimed. ∎

Proposition 6.44. *In $E^0(B(\Sigma_p \times \Delta)) = E^0[\![d, x]\!]/(df(d), [p^v](x))$ we have*

$$\psi_2(d) = d,\ \psi_2(c_p) = \prod_{k=0}^{p-1} (x +_F [k](w)) \text{ and } \psi_2(b_\alpha) = (p-1)! f(d) x^{\sum \alpha_i}.$$

Proof. The first claim is clear. For the second, let $V = \overline{\mathbb{F}}_l^p$ correspond to the representation $C_p \times \Delta \hookrightarrow GL_p(\overline{\mathbb{F}}_l)$ and $L = \overline{\mathbb{F}}_l$ to $C_p \rightarrowtail (\overline{\mathbb{F}}_l)^\times$. Then, since C_p acts on V by permuting the coordinates, it follows that V is the regular representation of C_p. Thus, standard representation theory (see, for example, [Ser77, 2.4]) gives $V \simeq \bigoplus_{k=0}^{p-1} L^k$ as $\overline{\mathbb{F}}_l$-representations of C_p. Now, let $M = \overline{\mathbb{F}}_l$ be the standard one-dimensional representation of $\Delta \simeq \mathbb{F}_q^\times \subseteq (\overline{\mathbb{F}}_l)^\times$. Then, since Δ acts diagonally on V, we have $V \simeq \bigoplus_{k=0}^{p-1} M \otimes_{\overline{\mathbb{F}}_l} L^k$ as $\overline{\mathbb{F}}_l$-representations of $C_p \times \Delta$. Hence we have

$$\begin{aligned} \mathrm{euler}_l(C_p \times \Delta \hookrightarrow GL_p(\overline{\mathbb{F}}_l)) &= \mathrm{euler}_l\left(\bigoplus_{k=0}^{p-1} M \otimes_{\overline{\mathbb{F}}_l} L^k\right) \\ &= \prod_{k=0}^{p-1} \mathrm{euler}_l(M \otimes_{\overline{\mathbb{F}}_l} L^k)) \\ &= \prod_{k=0}^{p-1} (\mathrm{euler}_l(M) +_F \mathrm{euler}_l(L^k)) \\ &= \prod_{k=0}^{p-1} (x +_F [k](w)). \end{aligned}$$

Thus, $\psi_2(c_p) = \mathrm{euler}_l(\Sigma_p \times \Delta \hookrightarrow GL_p(\overline{\mathbb{F}}_l))$ is just the pullback of this class under the injective map $E^0(B\Sigma_p \times \Delta) \hookrightarrow E^0(BC_p \times \Delta)$, as required.

For the final statement, note first that $(\Sigma_p \times \Delta) \backslash N / T = 1$ and that $(\Sigma_p \times \Delta) \cap T = \Delta$.

Writing S for $\Sigma_p \times \Delta$ we can apply the properties from Lemma 4.61 to get

$$\begin{aligned}
\psi_2(b_\alpha) &= \operatorname{res}^N_S \operatorname{tr}^N_T(x_1^{\alpha_1} \dots x_p^{\alpha_p}) \\
&= \sum_{\sigma \in S \backslash N / T} \operatorname{tr}^S_{S\cap T} \operatorname{res}^T_{S \cap T}(\operatorname{conj}^*_\sigma(x_1^{\alpha_1} \dots x_p^{\alpha_p})) \\
&= \operatorname{tr}^S_\Delta \operatorname{res}^T_\Delta(x_1^{\alpha_1} \dots x_p^{\alpha_p}) \\
&= \operatorname{tr}^S_\Delta(x^{\sum \alpha_i}) \\
&= \operatorname{tr}^{\Sigma_p \times \Delta}_{1 \times \Delta}(1 \otimes x^{\sum \alpha_i}) \\
&= \operatorname{tr}^{\Sigma_p}_1(1) \otimes \operatorname{tr}^\Delta_\Delta(x^{\sum \alpha_i}) \\
&= \operatorname{tr}^{\Sigma_p}_1(1) \otimes x^{\sum \alpha_i}.
\end{aligned}$$

But, from Lemma 4.89, $\operatorname{tr}^{\Sigma_p}_1(1) = (p-1)! f(d)$ and the result follows. ☐

Proposition 6.45. *In $E^0(BA) = E^0[\![x]\!]/[p^{v+1}](x)$ we have $\psi_3(d) = -[p^v](x)^{p-1}$.*

Proof. Write χ and ξ for the embeddings $A \rightarrowtail \overline{\mathbb{F}}_l^\times$ and $C_p \rightarrowtail \overline{\mathbb{F}}_l^\times$ respectively. Note that under the map $\pi : N \to \Sigma_p$ we have $\pi(A) = C_p$ and we get a commutative diagram

$$\begin{tikzcd}
& \overline{\mathbb{F}}_l^\times & \\
A \arrow[ur, "\xi\circ\pi"] \arrow[rr, two heads, "\pi"] \arrow[d] & & C_p \arrow[ul, "\xi"'] \arrow[d] \\
N \arrow[rr, two heads, "\pi"] & & \Sigma_p
\end{tikzcd}$$

Now, writing $a = \gamma(a_v, 1, \dots, 1)$ for the usual generator of A we have $(\xi \circ \pi)(a) = \xi(\gamma) = a_1 = a_{v+1}^{p^v} = \chi(a)^{p^v}$ so that $\xi \circ \pi = \chi^{p^v}$. Thus, applying $E^0(B-)$ to the above diagram gives

$$\begin{tikzcd}
& E^0(B\overline{\mathbb{F}}_l^\times) \arrow[dl, "(\chi^{p^v})^*"'] \arrow[dr, "\xi^*"] & \\
E^0(BA) & & E^0(BC_p) \arrow[ll, "\pi^*"'] \\
E^0(BN) \arrow[u, "\psi_3"] & & E^0(B\Sigma_p) \arrow[ll, "\pi^*"'] \arrow[u]
\end{tikzcd}$$

Hence $\pi^*(w) = \pi^*(\mathrm{euler}_l(\xi)) = \mathrm{euler}_l(\xi \circ \pi) = \mathrm{euler}_l(\chi^{p^v}) = [p^v](\mathrm{euler}_l(\chi)) = [p^v](x)$. Thus, since d is a class in $E^0(B\Sigma_p)$ and $d \mapsto -w^{p-1}$ in $E^0(BC_p)$ we find that

$$\psi_3(d) = \pi^*(-w^{p-1}) = -\pi^*(w)^{p-1} = -[p^v](x)^{p-1}.$$

☐

Proposition 6.46. *In $E^0(BA) = E^0[\![x]\!]/[p^{v+1}](x)$ we have $\psi_3(c_p) = \prod_{k=0}^{p-1}[1 + kp^v](x)$.*

Proof. We have $\psi_3(c_p) = \mathrm{euler}_l(A \hookrightarrow GL_p(\overline{\mathbb{F}}_l)) = \prod_{k=0}^{p-1}[q^k](x) = \prod_{k=0}^{p-1}[1 + kp^v](x)$ using Proposition 6.32. ☐

Remark 6.47. We can give an explicit diagonalisation of the representation $A \hookrightarrow GL_p(\overline{\mathbb{F}}_l)$. For $k = 0, \dots, p-1$ let $\nu_k = (1, a_{v+1}^{1+kp^v}, a_{v+1}^{2(1+kp^v)}, \dots, a_{v+1}^{(p-1)(1+kp^v)})^T$. Then it turns out that

ν_k is an eigenvector for $a = \gamma(a_v, 1, \ldots, 1)$ with eigenvalue $a_{v+1}^{1+kp^v}$ since

$$a.\nu_k = \gamma. \begin{pmatrix} a_v \\ a_{v+1}^{1+kp^v} \\ \vdots \\ a_{v+1}^{(p-1)(1+kp^v)} \end{pmatrix} = \begin{pmatrix} a_{v+1}^{1+kp^v} \\ \vdots \\ a_{v+1}^{(p-1)(1+kp^v)} \\ a_{v+1}^{p} \end{pmatrix} = a_{v+1}^{1+kp^v}\nu_k$$

where we use the fact that $a_v = a_{v+1}^p = a_{v+1}^{p(1+kp^v)}$. Thus $\gamma(a_v, 1, \ldots, 1)$ has distinct eigenvalues $a_{v+1}^{1+kp^v}$ for $0 \leq k < p$ and hence is conjugate to the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & a_{v+1}^{1+p^v} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{v+1}^{1+(p-1)p^v} \end{pmatrix},$$

the form of which is to be expected by earlier diagonalisation results.

Proposition 6.48. *In $E^0(BA) = E^0[\![x]\!]/[p^{v+1}](x)$ we get $\psi_3(b_\alpha) = (p-1)!x^{\sum \alpha_i}\langle p\rangle([p^v](x))$.*

Proof. Using the double coset formula we have

$$\psi_3(b_\alpha) = \mathrm{res}_A^N \mathrm{tr}_T^N(x_1^{\alpha_1} \ldots x_p^{\alpha_p}) = \sum_{A\backslash N/T} \mathrm{tr}_{A\cap T}^A \mathrm{res}_{A\cap T}^T(\mathrm{conj}_\sigma^*(x_1^{\alpha_1} \ldots x_p^{\alpha_p})).$$

But it is not hard to see that $A\backslash N/T \simeq \Sigma_p/\langle\gamma\rangle$ and $A \cap T = \Delta_p$ and it follows that $\mathrm{res}_{A\cap T}^T(\mathrm{conj}_\sigma^*(x_1^{\alpha_1} \ldots x_p^{\alpha_p})) = \mathrm{res}_{\Delta_p}^T(\mathrm{conj}_\sigma^*(x_1^{\alpha_1} \ldots x_p^{\alpha_p})) = x^{\sum \alpha_i}$ for all $\sigma \in A\backslash N/T$. Thus we have

$$\begin{aligned} \psi_3(b_\alpha) &= \sum_{\Sigma_p/\langle\gamma\rangle} \mathrm{tr}_{\Delta_p}^A \mathrm{res}_{\Delta_p}^T(x_1^{\alpha_1} \ldots x_p^{\alpha_p}) \\ &= |\Sigma_p/\langle\gamma\rangle|.\mathrm{tr}_{\Delta_p}^A(x^{\sum \alpha_i}) \\ &= (p-1)!\mathrm{tr}_{\Delta_p}^A(x^{\sum \alpha_i}). \end{aligned}$$

Now, the map $\mathrm{res}_{\Delta_p}^A : E^0(BA) \to E^0(B\Delta_p)$ sends x to x and hence

$$\mathrm{tr}_{\Delta_p}^A(x) = \mathrm{tr}_{\Delta_p}^A(\mathrm{res}_{\Delta_p}^A(x)) = x.\mathrm{tr}_{\Delta_p}^A(1)$$

by Frobenius reciprocity. Writing $q : A \to A/\Delta_p$ for the quotient map and using the properties of the transfer map (Lemma 4.61) we get $\mathrm{tr}_{\Delta_p}^A(1) = q^*\mathrm{tr}_1^{A/\Delta_p}(1)$. But A/Δ_p is naturally identified with C_p (since $A \xrightarrow{\pi} C_p$ has kernel Δ_p) and thus we have

$$\mathrm{tr}_{\Delta_p}^A(1) = \pi^*\mathrm{tr}_1^{C_p}(1) = \pi^*(\langle p\rangle(w)).$$

But, as in the proof of Proposition 6.45, $\pi^*(w) = [p^v](x)$. Hence we have

$$\psi_3(b_\alpha) = (p-1)!\mathrm{tr}_{\Delta_p}^A(x^{\sum \alpha_i}) = (p-1)!x^{\sum \alpha_i}\mathrm{tr}_{\Delta_p}^A(1) = (p-1)!x^{\sum \alpha_i}\langle p\rangle([p^v](x)). \quad \square$$

Proposition 6.49. *Let $t \in E^0(BN)$ be as in Proposition 6.41. Then $\psi_1(t) = 0$, $\psi_2(t) = 0$ and $\psi_3(t) = [p^v](x)^p$.*

Proof. Since $t \in \ker(\beta)$ it follows straight away that its image in $E^0(BT)$ is zero, so that $\psi_1(t) = 0$. As in Proposition 6.44, the representation $C_p \times \Delta \rightarrowtail GL_p(\overline{\mathbb{F}}_l)$ is isomorphic to a diagonal representation $C_p \times \Delta \to ((\overline{\mathbb{F}}_l)^\times)^p$ sending γ to $(1, a_1, \ldots, a_1^{p-1})$ and mapping Δ along the diagonal. Thus in cohomology we find that the map $E^0(B((\overline{\mathbb{F}}_l)^\times)^p) = E^0[\![x_1, \ldots, x_p]\!] \to E^0(B(C_p \times \Delta)) = E^0[\![w, x]\!]/([p](w), [p^v](x))$ sends x_k to $x +_F [k-1](w)$. We now see that $\psi_2(t) = \prod_{k=0}^{p-1}[p^v](x +_F [k](w)) = \prod_{k=0}^{p-1}[p^v](x) +_F [p^v]([k](w)) = 0$. By the methods of Proposition 6.32, we have a diagram

$$\begin{tikzcd}
E^0(BGL_p(\overline{\mathbb{F}}_l)) \arrow[r, "\mathrm{res}"] \arrow[dr] & E^0(B\mathbb{F}_{q^p}^\times) = E^0(BA) \\
& E^0(B(\overline{\mathbb{F}}_l^\times)^p) \arrow[u]
\end{tikzcd}$$

where the vertical map sends x_i to $[q^{i-1}](x)$. Hence we see that $\psi_3(t) = \prod_{i=1}^{p}[p^v]([q^{i-1}](x)) = \prod_{i=1}^{p}[p^v]([1 + kp^v](x)) = \prod_{i=1}^{p}[p^v](x) = [p^v](x)^p$ since $[p^{v+1}](x) = 0$. $\square$

6.4.6 A system of maps

In this section we will look at the earlier system of maps,

$$\begin{tikzcd}
& & E^0(BGL_p(\mathbb{F}_q)) \arrow[ddl, bend right, "\beta"'] \arrow[d] \arrow[ddr, bend left] \arrow[dddrr, bend left, "\alpha"] & & \\
& & E^0(BN) \arrow[dl, "\psi_1"'] \arrow[d, "\psi_2"] \arrow[dr, "\psi_3"] & & \\
& E^0(BT) \arrow[dr] & E^0(B(\Sigma_p \times \Delta)) \arrow[d] & E^0(BA) \arrow[dl, "q_1"] \arrow[dr, "q_2"] & \\
& & E^0(B\Delta_p) & & D
\end{tikzcd}$$

Proposition 6.50. *The map $\mathbb{Q} \otimes E^0(BA) \longrightarrow \mathbb{Q} \otimes E^0(B\Delta_p) \times \mathbb{Q} \otimes D$ induced by q_1 and q_2 is an isomorphism.*

Proof. Note first that $q_1(x) = x$ so that q_1 and q_2 are really nothing more than reduction modulo $([p](x))$ and $(\langle p\rangle([p](x)))$ respectively. Now, $\langle p\rangle([p](x)) = p \bmod ([p](x))$ and it follows that $p \in ([p](x)) + (\langle p\rangle([p](x))$. Thus Corollary 2.22 applies and the result is immediate. $\square$

Corollary 6.51. *The maps q_1 and q_2 defined above are jointly injective.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
E^0(BA) \arrow[r, "(q_1, q_2)"] \arrow[d] & E^0(B\Delta_p) \times D \arrow[d] \\
\mathbb{Q} \otimes E^0(BA) \arrow[r, "\sim"] & \mathbb{Q} \otimes E^0(B\Delta_p) \times \mathbb{Q} \otimes D.
\end{tikzcd}$$

Since $E^0(BA)$ is free over E^0 it follows that the map $E^0(BA) \to \mathbb{Q} \otimes E^0(BA)$ is injective and a diagram chase shows that the top map is also injective. $\square$

Corollary 6.52. *The maps ψ_1, ψ_2 and $q_2 \circ \psi_3$ defined above are jointly injective.*

Proof. Suppose $z \in E^0(BN)$ with z maps to 0 in each of $E^0(BT)$, $E^0(B(\Sigma_p \times \Delta))$ and D. Then z maps to 0 in $E^0(B\Delta_p)$ under $\mathrm{res}^T_{\Delta_p}$ so that, by Corollary 6.51, z maps to 0 in $E^0(BA)$. Thus we have $\psi_1(z), \psi_2(z)$ and $\psi_3(z)$ all zero, whereby $z = 0$ by Proposition 6.40. $\square$

Recall, from Proposition 6.16, that the map $E^0(BGL_p(\mathbb{F}_q)) \to E^0(BA)$ lands in the Γ-invariants, where $\Gamma \simeq \mathbb{Z}/p$ acts on $E^0(BA)$ by $k.x = [q^k](x)$. It follows that ψ_3 lands in $E^0(BA)^\Gamma$ and $q_2 \circ \psi_3$ lands in D^Γ.

Remark 6.53. We can in fact see the above Γ-invariance explicitly by looking at the images of the generators of $E^0(BN)$ under the map $q_2 \circ \psi_3$. Note first that, in $E^0(BA)$, $k.[p](x) = [p]([q^k](x)) = [pq^k](x) = [p](x)$ since $q^k = 1 \bmod p^v$. Hence $[p](x)$ and also $(q_2 \circ \psi_3)(d) = -[p](x)^{p-1}$ are Γ-invariant. We have $\psi_3(b_\alpha) = (p-1)!x^{\sum \alpha_i}\langle p\rangle([p^v](x))$ which is zero mod $\langle p\rangle([p^v](x))$, and so clearly maps into D^Γ. Finally, $\psi_3(c_p) = \prod_{j=0}^{p-1}[q^j](x)$ so that $k.\psi_3(c_p) = \prod_{j=0}^{p-1}[q^{j+k}](x) = \psi_3(c_p)$ since $q^p = 1 \bmod p^{v+1}$. Hence the generators of $E^0(BN)$ all land in D^Γ, as expected.

Note next that there is an injective restriction map $E^0(B(\Sigma_p \times \Delta)) \to E^0(B(C_p \times \Delta))$, where the latter ring has a presentation $E^0[\![w, x]\!]/([p](w), [p^v](x))$ with $w = \mathrm{euler}_l(C_p \rightarrowtail \overline{\mathbb{F}}_l^\times)$ and $x = \mathrm{euler}_l(\Delta \rightarrowtail \overline{\mathbb{F}}_l^\times)$. We will write ψ_2' for the composite map

$$E^0(BN) \xrightarrow{\psi_2} E^0(B(\Sigma_p \times \Delta)) \to E^0(B(C_p \times \Delta)) \twoheadrightarrow E^0(B(C_p \times \Delta))/\langle p\rangle(w).$$

This allows us the following proposition which will prove useful to us later.

Proposition 6.54. *The maps*

$$\begin{tikzcd} & E^0(BN) \arrow[dl, "\psi_1"'] \arrow[d, "\psi_2'"] \arrow[dr, "q_2 \circ \psi_3"] & \\ E^0(BT) & \dfrac{E^0(B(C_p \times \Delta))}{\langle p\rangle(w)} & D^\Gamma \end{tikzcd}$$

are jointly injective.

Proof. Since $[p](w) = w\langle p\rangle(w)$, an application of the Chinese remainder theorem (in particular, Corollary 2.22) shows that

$$\mathbb{Q} \otimes \frac{E^0[\![w, x]\!]}{([p](w), [p^v](x))} \simeq \mathbb{Q} \otimes \frac{E^0[\![w, x]\!]}{(w, [p^v](x))} \times \mathbb{Q} \otimes \frac{E^0[\![w, x]\!]}{(\langle p\rangle(w), [p^v](x))},$$

whereby, since $E^0(B(C_p \times \Delta))$ is free over E^0, we have jointly injective maps $E^0(B(C_p \times \Delta)) \to E^0[\![x]\!]/[p^v](x) = E^0(B\Delta)$ and $E^0(B(C_p \times \Delta)) \to E^0(B(C_p \times \Delta))/\langle p\rangle(w)$. By an argument like that of Corollary 6.52 the result follows. $\square$

6.4.7 Applying the theory of level structures

We now investigate some properties of the ring $E^0(B(C_p \times \Delta))/\langle p\rangle(w)$ which we identify with $E^0[\![w, x]\!]/(\langle p\rangle(w), [p^v](x))$ as in the remarks preceding Proposition 6.54.

Lemma 6.55. *The ring $R = E^0[\![w]\!]/\langle p\rangle(w)$ is an integral domain.*

Proof. With the usual notation we have $\langle p\rangle(w) \sim g_1(w)/w$ which is monic and irreducible over the integral domain E^0 as in Lemma 6.22. $\square$

Lemma 6.56. *The element* $y = \prod_{k=0}^{p-1}(x +_F [k](w)) \in R[\![x]\!] = E^0[\![w,x]\!]/\langle p\rangle(w)$ *is a unit multiple of* $\prod_{k=0}^{p-1}(x - [k](w))$.

Proof. Note that $y = \prod_{k=0}^{p-1}(x +_F [k](w)) = \prod_{k=0}^{p-1}(x -_F [k](w))$. Then an application of Lemma 4.14 shows that $y \sim \prod_{k=0}^{p-1}(x - [k](w))$. $\square$

To proceed further we need to use the results of [Str97] which require some familiarity with the language of formal schemes and, in particular, level structures. The aforementioned paper gives a good account of the basic definitions and notations.

Write $X = \mathrm{spf}(E^0)$ so that $\mathbb{G} = \mathrm{spf}(E^0(\mathbb{C}P^\infty)) = \mathrm{spf}(E^0[\![x]\!])$ is a formal group over X. Let $X_R = \mathrm{spf}(R)$. Then $\mathbb{G}_R = \mathbb{G} \times_X X_R = \mathrm{spf}(R[\![x]\!])$ is a formal group over X_R. Put $\mathbb{D} = \mathrm{spf}\,(R[\![x]\!]/y)$ and note that $\mathbb{D}$ is a formal scheme over X_R.

Proposition 6.57. $\mathbb{D}$ *is a subgroup scheme of degree* p *of* $\mathbb{G}_R(1) = \ker(\times p : \mathbb{G}_R \longrightarrow \mathbb{G}_R)$ *and* y *is a coordinate on the quotient group* $\mathbb{G}_R/\mathbb{D}$.

Proof. As in [HKR00], let $pF(R)$ denote the elements $a \in R$ for which $[p](a) = 0$, which is a group under $+_F$. Define a group homomorphism $\phi : \mathbb{Z}/p \to pF(R)$ given by $k \mapsto [k](w)$. Then ϕ is injective (since $[k](w) \sim w$ which is non-zero for $k \neq 0 \bmod p$) and hence, using the terminology of [Str97, Proposition 26], is a level structure on $\mathbb{G}_R$. Noting that, as divisors on $\mathbb{G}_R$, we have $\mathbb{D} = \mathrm{spf}(R[\![x]\!]/\prod_{k=0}^{p-1}(x - [k](w))) = [\phi(\mathbb{Z}/p)]$ and so, by Proposition 32 and Corollary 33 in [Str97] we find that $\mathbb{D}$ is a subgroup scheme of $\mathbb{G}_R$ contained in $\mathbb{G}_R(1)$ and that $\prod_{k=0}^{p-1}(x -_F [k](w)) = y$ is a coordinate on $\mathbb{G}_R/\mathbb{D}$. $\square$

Remark 6.58. It may be helpful to have a translation of some consequences of this result into algebra. We now know that there is a well defined map $R[\![x]\!]/y \to R[\![x]\!]/y$ sending x mod y to $[p](x)$ mod y; in other words $\prod_{k=0}^{p-1}([p](x) - [k](w))$ is divisible by $\prod_{k=0}^{p-1}(x - [k](w))$ in $R[\![x]\!]$. This should not be too difficult to believe, as the former is divisible by $[p](x)$ and $[p]([k](w)) = [k]([p](w)) = 0$ in R so that $(x - [k](w))$ is certainly a factor; the above result tells us that $\prod_{k=0}^{p-1}(x - [k](w))$ is also a factor. The remarks about the quotient group $\mathbb{G}_R/\mathbb{D}$ give us a subring $\mathcal{O}_{\mathbb{G}_R/\mathbb{D}} = R[\![y]\!]$ of $\mathcal{O}_{\mathbb{G}_R} = R[\![x]\!]$ with favourable properties, as we will see below.

Proposition 6.59. *Let* $y = \prod_{k=0}^{p-1}(x +_F [k](w)) \in R[\![x]\!]$, *as above. Then there is a unique power-series* $h \in R[\![y]\!]$ *such that* $h = [p^v](x)$ *in* $R[\![x]\!]$.

Proof. Since $\mathbb{D}$ is contained in $\mathbb{G}_R(1)$, the map $\mathbb{G}_R \xrightarrow{\times p^v} \mathbb{G}_R$ factors through $\mathbb{G}_R/\mathbb{D}$. Thus there is a map $\psi : \mathbb{G}_R/\mathbb{D} \to \mathbb{G}_R$ making the diagram below commute.

$$\begin{tikzcd} \mathbb{G}_R \arrow[rr, "\times p^v"] \arrow[dr] & & \mathbb{G}_R \\ & \mathbb{G}_R/\mathbb{D}. \arrow[ur, "\psi"'] & \end{tikzcd}$$

Put $h(y) = \psi^*(x) \in \mathcal{O}_{\mathbb{G}_R/\mathbb{D}} = R[\![y]\!]$. Then $h(y) = [p^v](x)$ in $\mathcal{O}_{\mathbb{G}_R} = R[\![x]\!]$, as required. Uniqueness is immediate since $\mathcal{O}_{\mathbb{G}_R/\mathbb{D}}$ is a subring of $\mathcal{O}_{\mathbb{G}_R}$. $\square$

We explore some of the properties of the power-series h defined above, but first need a couple of lemmas.

Lemma 6.60. *Let A be a ring, let G be a finite group acting on A and suppose that $|G|$ is invertible in A. Then for any $a \in A^G$ there is an action of G on A/aA and there is a G-equivariant isomorphism $A^G/aA^G \simeq (A/aA)^G$.*

Proof. It is clear that the composite $f : A^G \rightarrowtail A \to A/aA$ has image contained in $(A/aA)^G$. Given any $r + aA \in (A/aA)^G$ we find that $f(\frac{1}{|G|}\sum_{g\in G} g.r) = \frac{1}{|G|}\sum_{g\in G}(g.r + aA) = \frac{|G|}{|G|}(r + aA) = r + aA$ so that $f : A^G \to (A/aA)^G$ is surjective. It is clear that the kernel of the map is just aA^G and the result follows. ∎

Lemma 6.61. *There is an action of $(\mathbb{Z}/p)^\times$ on R given by $k.w = [k](w)$ and $\langle p\rangle(w)$ is $(\mathbb{Z}/p)^\times$-invariant. Further, the element $d = \prod_{k=1}^{p-1}[k](w)$ generates $R^{(\mathbb{Z}/p)^\times}$ over E^0 and $R^{(\mathbb{Z}/p)^\times} = E^0[\![d]\!]/f(d)$, where $f(d) = \langle p\rangle(x)$ as an element of $R[\![x]\!]$.*

Proof. As is Proposition 4.88, we know that $(E^0[\![w]\!]/[p](w))^{(\mathbb{Z}/p)^\times} = E^0[\![d]\!]/df(d)$. Then an application of Lemma 6.60 shows that $(E^0[\![w]\!]/\langle p\rangle(w))^{(\mathbb{Z}/p)^\times} = E^0[\![d]\!]/f(d)$. ∎

Corollary 6.62. *With h as in Proposition 6.59 and $(\mathbb{Z}/p)^\times$ acting on R as above, we have $h \in R[\![y]\!]^{(\mathbb{Z}/p)^\times} = E^0[\![d, y]\!]/f(d)$.*

Proof. We know that the inclusion $R[\![y]\!] \rightarrowtail R[\![x]\!]$ maps h to $[p^v](x)$, which is clearly invariant under the action of $(\mathbb{Z}/p)^\times$ on R. Hence $h \in R[\![x]\!]^{(\mathbb{Z}/p)^\times} \cap R[\![y]\!] = R[\![y]\!]^{(\mathbb{Z}/p)^\times}$. ∎

We will write $h = h(d, y) \in E^0[\![d, y]\!]/f(d)$ thought of as a power-series in d and y. Note that there is a well defined element $dh(d, y) \in E^0[\![d, y]\!]/df(d)$ and hence a well defined element $dh(d, c_p) \in E^0(BN)$.

Proposition 6.63. *With h as in Proposition 6.59 we have $t + dh(d, c_p) = 0$ in $E^0(BN)$.*

Proof. We check that $t + dh(d, c_p)$ maps to zero in each of $E^0(BT)$, $E^0(B(C_p \times \Delta))/\langle p\rangle(w)$ and D^Γ and conclude that it must be zero in $E^0(BN)$ by Corollary 6.54. We use the results of Proposition 6.42. In $E^0(BT)$ we have both t and d mapping to 0 so that $t + dh(d, a) \mapsto 0$. In $E^0(B(C_p \times \Delta))/\langle p\rangle(w) = E^0[\![w, x]\!]/(\langle p\rangle(w), [p^v](x))$ we have

$$t + dh(d, c_p) \mapsto 0 + dh\left(d, \prod_{k=0}^{p-1}(x +_F [k](w))\right) = d[p^v](x) = 0.$$

We are left to consider the image in D^Γ. There is a well-defined map $E^0[\![w, x]\!]/\langle p\rangle(w) \to D^\Gamma$ given by $w \mapsto [p^v](x)$ and $x \mapsto x$. Then the identity $h\left(-w^{p-1}, \prod_{k=0}^{p-1}(x +_F [k](w))\right) = [p^v](x)$ gives $h\left(-[p^v](x)^{p-1}, \prod_{k=0}^{p-1}[1 + kp^v](x)\right) = [p^v](x)$ in D^Γ. Thus we have

$$t + dh(d, c_p) \mapsto [p^v](x)^p + (-[p^v](x)^{p-1})h\left(-[p^v](x)^{p-1}, \prod_{k=0}^{p-1}[1 + kp^v](x)\right) = 0$$

in D^Γ and we are done. ∎

Corollary 6.64. *In $K^0(BN)$ we have $t = c_p^{p^{nv}}$ and $c_p^{p^{nv}} + dh(d, c_p) = 0$.*

Proof. We have $t = \prod_{i=1}^{p}[p^v](x_i) = \prod_{i=1}^{p} x_i^{p^{nv}} \mod (p, u_1, \ldots, u_{n-1})$. ∎

We will be interested in decoding this relation a bit further, and the following will help.

Lemma 6.65. *In $K^0 \otimes_{E^0} E^0[\![w,s]\!]/\langle p\rangle(w)$ we have $h(-w^{p-1}, s) = s^{p^{nv-1}}$ mod w. Hence, $h(0,s) = s^{p^{nv-1}} \in \mathbb{F}_p[\![s]\!]$.*

Proof. The identity $h(-w^{p-1}, \prod_{k=0}^{p-1} x +_F [k](w)) = [p^v](x)$ in $E^0[\![w,x]\!]/\langle p\rangle(w)$ read modulo $(p, u_1, \ldots, u_{n-1})$ gives $h(-w^{p-1}, \prod_{k=0}^{p-1} x +_F [k](w)) = x^{p^{nv}}$. Then, modulo w, we get $h(0, x^p) = x^{p^{nv}}$ and hence $h(0,s) = s^{p^{nv-1}}$. $\square$

In Section 6.4.9 we will need these results to get an idea of the structure of the kernel of the map $\beta : E^0(BGL_p(\mathbb{F}_q)) \to E^0(BT)^{\Sigma_p}$. In particular, they will be crucial in proving that a certain class in $K^0(BN)$ is non-zero.

6.4.8 The kernels of α and β

In a similar fashion to the results of Section 6.4.6, we aim to prove joint injectivity of the maps α and β.

Lemma 6.66. *Let $g \in GL_d(K)$. Then there exists a permutation $\rho \in \Sigma_d$ such that $g_{i\rho(i)} \neq 0$ for all i.*

Proof. Label the rows of g as $r_1, \ldots, r_d$. Then, by considering the expansion of the determinant along r_1, there must be a non-zero entry r_{1j} such that the resultant matrix formed by deleting row 1 and column j has non-zero determinant, and is therefore invertible. Put $\rho(1) = j$ and continue inductively to get a well-defined permutation $\rho \in \Sigma_d$ with the required property. $\square$

Lemma 6.67. *Let $\mathcal{A}'$ be the full subcategory of $\mathcal{A}(G)_{(p)}$ with objects A, $T_{(p)}$ and Δ_p. Then*

$$\lim_{H \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^0(BH) = \lim_{H \in \mathcal{A}'} \mathbb{Q} \otimes E^0(BH).$$

Proof. There is a unique map $\lim_{H \in \mathcal{A}(G)_{(p)}} \mathbb{Q} \otimes E^0(BH) \to \lim_{H \in \mathcal{A}'} \mathbb{Q} \otimes E^0(BH)$ commuting with the arrows, by abstract category theory. We construct an inverse.

Recall from Proposition 3.22 that any abelian p-subgroup of $GL_p(\mathbb{F}_q)$ is sub-conjugate to either A or T (or both). Write $L = \lim_{H \in \mathcal{A}'} \mathbb{Q} \otimes E^0(BH)$. Note that the structure maps $L \to \mathbb{Q} \otimes E^0(BA)$ and $L \to \mathbb{Q} \otimes E^0(BT_{(p)})$ land in the invariant subrings under the action of the relevant normaliser. We consider abelian p-subgroups H of $GL_p(\mathbb{F}_q)$.

If H is cyclic of order p^{v+1} then H must be conjugate to A. Further, given two such isomorphisms $g_1 H g_1^{-1} = A = g_2 H g_2^{-1}$ the diagram

$$\begin{tikzcd} A \arrow[d, "\mathrm{conj}_{g_2 g_1^{-1}}"'] & H \arrow[l, "\mathrm{conj}_{g_1}"'] \arrow[dl, "\mathrm{conj}_{g_2}"] \\ A & \end{tikzcd}$$

commutes. Thus $g_2 g_1^{-1} \in N_{GL_p(\mathbb{F}_q)}(A)$ and both maps $L \to \mathbb{Q} \otimes E^0(BA) \to \mathbb{Q} \otimes E^0(BH)$ are equal. Hence we have a uniquely defined map $L \to \mathbb{Q} \otimes E^0(BH)$. Further, given any map $\mathrm{conj}_g : H_1 \to H_2$ of such subgroups in $\mathcal{A}(G)_{(p)}$ it is clear from similar reasoning that we have a commuting diagram

$$\begin{tikzcd}
 & & \mathbb{Q}\otimes E^0(BH_1) \arrow[dd, "\mathrm{conj}_g^*"] \\
L \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \mathbb{Q}\otimes E^0(BA) \arrow[ur] \arrow[dr] & \\
 & & \mathbb{Q}\otimes E^0(BH_2).
\end{tikzcd}$$

Any other $H \leqslant GL_p(\mathbb{F}_q)$ which is sub-conjugate to A must be sub-conjugate to $\Delta_p \subseteq A$. In particular, since Δ is central in $GL_p(\mathbb{F}_q)$, we find that $H \subseteq \Delta_p$ and, for any $g \in GL_p(\mathbb{F}_q)$, the conjugation map induced by g is the identity on H. Hence we have a uniquely defined map $L \to \mathbb{Q}\otimes E^0(BA) \to \mathbb{Q}\otimes E^0(B\Delta_p) \to \mathbb{Q}\otimes E^0(BH)$ which respects any arrow in $\mathcal{A}(G)_{(p)}$.

We are left with the case that H is subconjugate to T. First, suppose that $H \subseteq T$ and let $h = (h_1, \ldots, h_p) \in H$. Then $ghg^{-1} = k$ for some $k = (k_1, \ldots, k_p) \in T$. Letting $g = (g_{ij})$ we get equations $g_{ij}h_j = k_i g_{ij}$ for all i, j. Hence $g_{ij}(h_j - k_i) = 0$. But, by Lemma 6.66, there is a permutation $\rho \in \Sigma_p$ with $g_{i\rho(i)} \in \mathbb{F}_q^\times$ for all i. Thus, for each i we have $h_{\rho(i)} = k_i$ and so $ghg^{-1} = (h_{\rho(1)}, \ldots, h_{\rho(p)})$. It follows that the map $\mathrm{conj}_g : H \to T$ corresponds to permutation by ρ and hence extends to a map $T \to T$ induced by an element of $N_{GL_p(\mathbb{F}_q)}(T)$.

Now, given any H subconjugate to $T_{(p)}$ it follows that the map $L \to \mathbb{Q}\otimes E^0(BT_{(p)}) \to \mathbb{Q}\otimes E^0(BH)$ is independent of the choice of conjugating element and, further, that any arrow in $\mathcal{A}(G)_{(p)}$ commutes with these maps.

Thus, we conclude that given any arrow $H \to K$ in $\mathcal{A}(G)_{(p)}$ we have maps $L \to \mathbb{Q}\otimes E^0(BH)$ and $L \to \mathbb{Q}\otimes E^0(BK)$ which commute with the arrow. Hence we get a well defined map $L \to \lim_{H\in\mathcal{A}(G)_{(p)}} \mathbb{Q}\otimes E^0(BH)$ which is necessarily inverse to the map at the start of the proof by abstract category theory. $\square$

Proposition 6.68. *The map $\mathbb{Q}\otimes E^0(BGL_p(\mathbb{F}_l)) \longrightarrow \mathbb{Q}\otimes E^0(BT)^{\Sigma_p} \times \mathbb{Q}\otimes D^\Gamma$ induced by α and β is an isomorphism.*

Proof. Writing $G = GL_p(\mathbb{F}_q)$, by Proposition 4.70 we have an isomorphism

$$\mathbb{Q}\otimes E^0(BGL_p(\mathbb{F}_q)) \xrightarrow{\sim} \lim_{H\in\mathcal{A}(G)_{(p)}} \mathbb{Q}\otimes E^0(BH).$$

But, by Lemma 6.67, the right-hand side simplifies to $\lim_{H\in\mathcal{A}'} \mathbb{Q}\otimes E^0(BH)$. Thus, using Proposition 3.10 and Lemma 3.23 we are left with a pullback

$$\begin{tikzcd}
\mathbb{Q}\otimes E^0(BGL_p(\mathbb{F}_q)) \arrow[r] \arrow[d] & \mathbb{Q}\otimes E^0(BA)^\Gamma \arrow[d] \\
\mathbb{Q}\otimes E^0(BT_{(p)})^{\Sigma_p} \arrow[r] & \mathbb{Q}\otimes E^0(B\Delta_p).
\end{tikzcd}$$

From Proposition 6.50 we know that $\mathbb{Q}\otimes E^0(BA) \simeq \mathbb{Q}\otimes E^0(B\Delta_p) \times \mathbb{Q}\otimes D$. But the action of Γ is trivial on $E^0(B\Delta_p)$ so we get $\mathbb{Q}\otimes E^0(BA)^\Gamma \simeq \mathbb{Q}\otimes E^0(B\Delta_p) \times \mathbb{Q}\otimes D^\Gamma$. Since $E^0(BT)^{\Sigma_p} \to E^0(B\Delta_p)$ is surjective, the result follows. $\square$

Corollary 6.69. *The maps*

$$\begin{tikzcd} & E^0(BGL_p(\mathbb{F}_q)) \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ E^0(BT)^{\Sigma_p} & & D^\Gamma \end{tikzcd}$$

are jointly injective.

Proof. Since $E^0(BGL_p(\mathbb{F}_q))$ is free over E^0 the result follows analogously to the proof of Corollary 6.51. □

Corollary 6.70. *$rank_{E^0}(E^0(BGL_p(\mathbb{F}_l))) = rank_{E^0}(E^0(BT)^{\Sigma_p}) + rank_{E^0}(D^\Gamma)$. Hence we have $rank_{E^0}(E^0(BGL_p(\mathbb{F}_q))) = rank_{E^0}(\ker(\alpha)) + rank_{E^0}(\ker(\beta))$.*

Proof. Each of the rings in question is free over E^0 so

$$\begin{aligned} \mathrm{rank}_{E^0}(E^0(BGL_p(\mathbb{F}_l))) &= \mathrm{rank}_{\mathbb{Q}\otimes E^0}(\mathbb{Q}\otimes E^0(BGL_p(\mathbb{F}_l))) \\ &= \mathrm{rank}_{\mathbb{Q}\otimes E^0}(\mathbb{Q}\otimes_{E^0} E^0(BT)^{\Sigma_p}) + \mathrm{rank}_{\mathbb{Q}\otimes E^0}(\mathbb{Q}\otimes_{E^0} D^\Gamma) \\ &= \mathrm{rank}_{E^0}(E^0(BT)^{\Sigma_p}) + \mathrm{rank}_{E^0}(D^\Gamma). \end{aligned}$$

For the final remark, note that

$$\mathrm{rank}_{E^0}(E^0(BGL_p(\mathbb{F}_q))) = \mathrm{rank}_{E^0}(\ker(\beta)) + \mathrm{rank}_{E^0}(\mathrm{image}(\beta))$$

and $\mathrm{image}(\beta) = E^0(BT)^{\Sigma_p}$. It follows that $\mathrm{rank}_{E^0}(\ker(\beta)) = \mathrm{rank}_{E^0}(D^\Gamma)$ and similarly that $\mathrm{rank}_{E^0}(\ker(\alpha)) = \mathrm{rank}_{E^0}(E^0(BT)^{\Sigma_p})$. □

Recall that $I = E^0(BGL_p(\mathbb{F}_q))t$ was the ideal of $E^0(BGL_p(\mathbb{F}_q))$ generated by t so that, by Proposition 6.41, we have $I \subseteq \ker(\beta)$. We suspect that $I = \ker(\beta)$ and work towards proving the reverse inclusion.

Lemma 6.71. *We have $\ker(\alpha)\cap\ker(\beta) = 0$ and hence $\ker(\alpha).\ker(\beta) = 0$.*

Proof. By Corollary 6.69 the map $(\alpha,\beta) : E^0(BLG_p(\mathbb{F}_q)) \to D^\Gamma \times E^0(BT)^{\Sigma_p}$ is injective. If $a \in \ker(\alpha)\cap\ker(\beta)$ then $\alpha(a) = \beta(a) = 0$ whereby $a \in \ker(\alpha,\beta) = 0$. The second claim follows as $\ker(\alpha).\ker(\beta) \subseteq \ker(\alpha)\cap\ker(\beta)$. □

Corollary 6.72. *The identification $E^0(BGL_p(\mathbb{F}_q))/\ker(\alpha) \simeq D^\Gamma$ makes I into a free rank one module over D^Γ and hence a free E^0-module of rank N.*

Proof. Since $t \in \ker(\beta)$ we have $\ker(\alpha)t = 0$ and it follows that

$$I = E^0(BGL_p(\mathbb{F}_q))t = \left(\frac{E^0(BGL_p(\mathbb{F}_q))}{\ker(\alpha)}\right) t \simeq D^\Gamma t.$$

Now, as I is generated by one element over D^Γ it is sufficient to show that I is torsion free. Take $0 \neq s \in D^\Gamma$. Then $\alpha(s.t) = s\alpha(t) \sim s[p^v](x)^p$. Since $[p^v](x)$ divides $\langle p\rangle([p^v](x)) - p$ we see that $[p^v](x)$ divides p in D. Thus $s[p^v](x)^p$ divides sp^p, which is non-zero as D is free over E^0. Hence $\alpha(s.t) \neq 0$ and $s.t \neq 0$, as required. The final statement is immediate from Proposition 6.29. □

Lemma 6.73. *The ideal* $\ker(\beta)$ *is an* E^0*-module summand in* $E^0(BGL_p(\mathbb{F}_q))$ *and is free of rank* N*. Similarly,* $\ker(\alpha)$ *is a free* E^0*-summand in* $E^0(BGL_p(\mathbb{F}_q))$ *of rank* $\frac{(p^{nv}+p-1)!}{p!(p^{nv}-1)!}$.

Proof. Since each of $E^0(BGL_p(\mathbb{F}_q))$ and $E^0(BT)^{\Sigma_p}$ is free over E^0 it follows that the short exact sequence $0 \to \ker(\beta) \to E^0(BGL_p(\mathbb{F}_q)) \to E^0(BT)^{\Sigma_p} \to 0$ splits. Thus $\ker(\beta)$ is a summand in $E^0(BGL_p(\mathbb{F}_q))$ and hence is projective. But all projective modules over local rings are free (see [Kap58, Theorem 2]). Hence $\ker(\beta)$ is free. By Proposition 6.70 we have $\mathrm{rank}_{E^0}(E^0(BGL_p(\mathbb{F}_q))) = \mathrm{rank}_{E^0}(E^0(BT)^{\Sigma_p}) + \mathrm{rank}_{E^0}(D^\Gamma)$ and it follows that $\mathrm{rank}_{E^0}(\ker(\beta)) = \mathrm{rank}_{E^0}(D^\Gamma) = N$.

Since D^Γ is also free over E^0 an exactly analogous argument will prove that $\ker(\alpha)$ is a free E^0-summand. The rank of $\ker(\alpha)$ is the same as that of $\mathrm{rank}_{E^0}(E^0(BT)^{\Sigma_p})$ which, by Proposition 2.9, is equal to the size of the set $\{\beta \in \mathbb{N}^p \mid 0 \le \beta_1 + \ldots + \beta_p < p^{nv}\}$. Standard combinatorics (p markers in $p^{nv}+p-1$ slots) gives this as $\binom{p^{nv}+p-1}{p} = \frac{(p^{nv}+p-1)!}{p!(p^{nv}-1)!}$. ☐

Corollary 6.74. *We have* $\ker(\alpha) = \mathit{ann}(\ker(\beta))$ *and* $\ker(\beta) = \mathit{ann}(\ker(\alpha))$.

Proof. Firstly note that $\ker(\alpha) \subseteq \mathrm{ann}(\ker(\beta))$ and $\ker(\beta) \subseteq \mathrm{ann}(\ker(\alpha))$ by Lemma 6.71. By [Str00] we know that $E^0(BGL_p(\mathbb{F}_q))$ has duality over E^0. Thus we can apply Corollary 2.24 to see that both of $\mathrm{ann}(\ker(\alpha))$ and $\mathrm{ann}(\ker(\beta))$ are summands in $E^0(BGL_p(\mathbb{F}_q))$ and hence free. But $\mathrm{rank}_{E^0}(\mathrm{ann}(\ker(\beta))) = \mathrm{rank}_{E^0}(E^0(BGL_p(\mathbb{F}_q))) - \mathrm{rank}_{E^0}(\ker(\beta))$ and the latter is just $\mathrm{rank}_{E^0}(\ker(\alpha))$ using Proposition 6.70. Thus $\ker(\alpha) \subseteq \mathrm{ann}(\ker(\beta))$ is an inclusion of free summands of the same rank and so is an equality. The same argument shows that $\ker(\beta) = \mathrm{ann}(\ker(\alpha))$. ☐

6.4.9 Studying $\ker(\beta)$ more closely

To proceed further we apply the functor $K^0 \otimes_{E^0} -$ or, equivalently, work modulo the maximal ideal $(p, u_1, \ldots, u_{n-1})$. We have the following commutative diagram.

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & \ker(\beta) \arrow[r] \arrow[d] & E^0(BGL_p(\mathbb{F}_q)) \arrow[r, two heads, "\alpha"] \arrow[d] & D^\Gamma \arrow[d] \\
K^0 \otimes_{E^0} I \arrow[r] & K^0 \otimes_{E^0} \ker(\beta) \arrow[r] & K^0 \otimes_{E^0} E^0(BGL_p(\mathbb{F}_q)) \arrow[r, two heads] & K^0 \otimes_{E^0} D^\Gamma
\end{tikzcd}$$

We know that $K^0 \otimes_{E^0} E^0(BGL_p(\mathbb{F}_q)) = K^0(BGL_p(\mathbb{F}_q))$ and aim to understand the remainder of the bottom row. Recall that we defined N to be $(p^{n(v+1)} - p^{nv})/p$.

Proposition 6.75. *With* y *as in Proposition 6.29 we have* $K^0 \otimes_{E^0} D^\Gamma \simeq \mathbb{F}_p[\![y]\!]/y^N$.

Proof. Modulo $(p, u_1, \ldots, u_{n-1})$ we know that $[p^m](x) = x^{p^{nm}}$ for all m and hence that $\langle p\rangle([p^v](x)) = x^{p^{n(v+1)} - p^{nv}}$. Thus

$$K^0 \otimes_{E^0} D = \frac{K^0 \otimes_{E^0} E^0[\![x]\!]}{K^0 \otimes_{E^0} (\langle p\rangle([p](x)))} = \mathbb{F}_p[\![x]\!]/(x^{p^{n(v+1)} - p^{nv}}).$$

Further, $K^0 \otimes_{E^0} D^\Gamma$ is the subring of $K^0 \otimes_{E^0} D$ generated by $y \sim x^p$, so that $K^0 \otimes_{E^0} D^\Gamma = \mathbb{F}_p[\![y]\!]/y^N$, as claimed. ☐

Lemma 6.76. $\ker(K^0 \otimes \beta)$ *is a module over* $K^0 \otimes_{E^0} D^\Gamma$.

Proof. By Lemma 2.17, the maps $K^0 \otimes_{E^0} \ker(\alpha) \to \ker(K^0 \otimes \alpha)$ and $K^0 \otimes_{E^0} \ker(\beta) \to \ker(K^0 \otimes \beta)$ are both surjective. Take $a \in \ker(K^0 \otimes \alpha)$. We can lift a first to $K^0 \otimes_{E^0} \ker(\alpha)$ and then to $\ker(\alpha)$. Choose such a lift, $\tilde{a} \in \ker(\alpha)$ say. Similarly, given any $b \in \ker(K^0 \otimes_{E^0} \beta)$ we can choose a lift $\tilde{b} \in \ker(\beta)$. Then $\tilde{a}.\tilde{b} = 0$ in $E^0(BGL_p(\mathbb{F}_q))$ by Lemma 6.71 so that $a.b = 0$ in $K^0(BGL_p(\mathbb{F}_q))$. It follows that $\ker(K^0 \otimes_{E^0} \beta)$ is a module over $K^0(BGL_p(\mathbb{F}_q))/\ker(K^0 \otimes \alpha) = K^0 \otimes_{E^0} D^\Gamma$. □

Corollary 6.77. *$K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}}$ is a module over $K^0 \otimes_{E^0} D^\Gamma$.*

Proof. Since $t \in \ker(\beta)$ and $t = c_p^{p^{nv}}$ modulo $(p, u_1, \ldots, u_{n-1})$ it follows that $c_p^{p^{nv}}$ maps to zero under $K^0 \otimes \beta$. Thus $K^0(BGL_p(\mathbb{F}_q))c_p^{p^n} \subseteq \ker(K^0 \otimes \beta)$ and so is annihilated by $\ker(K^0 \otimes \alpha)$. The result follows. □

To understand the structure of $K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}}$ as a $K^0 \otimes_{E^0} D^\Gamma$-module we will need an understanding of the nilpotency of $c_p^{p^{nv}}$ in $K^0(BN)$.

Lemma 6.78. *In $K^0(BN)$ we have $c_p^{p^{nv+i}} \sim c_p^{p^{nv-1}} d^{1+p+\ldots+p^i}$ modulo $d^{1+p+\ldots+p^i+1}$ for each $i \geq 0$.*

Proof. We proceed by induction. For $i = 0$ we have $c_p^{p^{nv}} \sim dh(d, c_p)$ by Corollary 6.64. From Lemma 6.65 we get $h(d, c_p) = c_p^{p^{nv-1}} \bmod d$ so that $dh(d, c_p) = c_p^{p^{nv-1}} d \bmod d^2$, as required. Supposing $c_p^{p^{nv+k}} \sim c_p^{p^{nv-1}} d^{1+p+\ldots+p^k} \bmod d^{1+p+\ldots+p^k+1}$ write

$$c_p^{p^{nv+k}} = uc_p^{p^{nv-1}} d^{1+p+\ldots+p^k} + d^{1+p+\ldots+p^k+1} s$$

for some unit u and some s. Then, raising to the power p (a mod-p automorphism) we have

$$c_p^{p^{nv+k+1}} = u^p c_p^{p^{nv}} d^{p+p^2+\ldots+p^{k+1}} + d^{p+p^2+\ldots+p^{k+1}+p} s^p.$$

Thus, modulo $d^{1+p+\ldots+p^{k+1}+1}$, we have $c_p^{p^{nv+k+1}} \sim c_p^{p^{nv}} d^{p+p^2+\ldots+p^{k+1}}$ since $p \geq 2$. But $c_p^{p^{nv}} \sim c_p^{p^{nv-1}} d \bmod d^2$ so that $c_p^{p^{nv}} d^{p+p^2+\ldots+p^{k+1}} \sim c_p^{p^{nv-1}} d^{1+p+\ldots+p^{k+1}} \bmod d^{p+\ldots+d^{p^{k+1}}+2}$. Hence $c_p^{p^{nv+k+1}} \sim c_p^{p^{nv-1}} d^{1+p+\ldots+p^{k+1}} \bmod d^{1+p+\ldots+p^{k+1}+1}$, completing the inductive step. □

Lemma 6.79. *In $K^0(BN)$ we have $c_p^{p^{n(v+1)-1}} \sim c_p^{p^{nv-1}} d^{(p^n-1)/(p-1)}$, which is non-zero.*

Proof. Put $i = n-1$ in Lemma 6.78 to get the result $c_p^{p^{n(v+1)-1}} \sim c_p^{p^{nv-1}} d^{1+p+\ldots+p^{n-1}} \bmod d^{1+p+\ldots+p^{n-1}+1}$. But we have $d^{1+p+\ldots+p^{n-1}+1} = d^{(p^n-1)/(p-1)+1} = 0$ in $K^0(BN)$ (since, by tensoring with K^0, $K^0(B\Sigma_p) \simeq \mathbb{F}_p[\![d]\!]/d^{(p^n-1)/(p-1)+1}$). Thus we get

$$c_p^{p^{n(v+1)-1}} \sim c_p^{p^{nv-1}} d^{1+p+\ldots+p^{n-1}} = c_p^{p^{nv-1}} d^{(p^n-1)/(p-1)}$$

in $K^0(BN)$. The right-hand side is a basis element for $K^0(BN)$ over K^0 so is non-zero. □

Proposition 6.80. *We have $c_p^{N+p^{nv}-1} \neq 0$ in $K^0(BGL_p(\mathbb{F}_q))$.*

Proof. By Lemma 6.79 we have $c_p^{p^{n(v+1)-1}} \sim c_p^{p^{nv-1}} d^{(p^n-1)/(p-1)}$. Multiplying both sides by $c_p^{p^{nv}-p^{nv-1}-1}$ then gives $c_p^{N+p^{nv}-1} \sim c_p^{p^{nv}-1} d^{(p^n-1)/(p-1)} \neq 0$. Thus $c_p^{N+p^{nv}-1}$ is non-zero in $K^0(BN)$. □

Proposition 6.81. *The ideal $K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}}$ is free of rank 1 over $K^0 \otimes_{E^0} D^\Gamma = \mathbb{F}_p[\![y]\!]/y^N$ and hence has dimension N as a vector space over $\mathbb{F}_p$.*

Proof. Since $M = K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}}$ is generated over $\mathbb{F}_p[\![y]\!]/y^N$ by one element, namely $c_p^{p^{nv}}$, it follows that $M \simeq \mathbb{F}_p[\![y]\!]/y^m$ for some $m \leq N$, that is $y^m.c_p^{p^{nv}} = 0$ for some m. Since α maps c_p to y we have $y^{N-1}.c_p^{p^{nv}} \neq 0$ if and only if $c_p^{N-1}c_p^{p^{nv}} \neq 0$. By Corollary 6.80, the latter holds so that $M \simeq \mathbb{F}_p[\![y]\!]/y^N$ is free over $K^0 \otimes_{E^0} D^\Gamma$. □

Lemma 6.82. *The induced map $K^0 \otimes_{E^0} I \to K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}}$ is an isomorphism.*

Proof. That the map is surjective is immediate from Corollary 2.18. Using Corollary 6.72 we see that $K^0 \otimes_{E^0} I$ is an $\mathbb{F}_p$-vector space of dimension $N = \dim_{\mathbb{F}_p}(K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}})$. Thus the map is an isomorphism. □

Lemma 6.83. *The induced map $K^0 \otimes_{E^0} I \to K^0 \otimes_{E^0} \ker(\beta)$ is an isomorphism.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
K^0 \otimes_{E^0} I \arrow[r] \arrow[dr, "\sim"'] & K^0 \otimes_{E^0} \ker(\beta) \arrow[r] & \ker(K^0 \otimes \beta) \\
& K^0(BGL_p(\mathbb{F}_q)).c_p^{p^{nv}} \arrow[ur, tail] &
\end{tikzcd}$$

Since the composite $K^0 \otimes_{E^0} I \xrightarrow{\sim} K^0(BGL_p(\mathbb{F}_q))c_p^{p^{nv}} \rightarrowtail \ker(K^0 \otimes \beta)$ is injective we see that the map $K^0 \otimes_{E^0} I \to K^0 \otimes_{E^0} \ker(\beta)$ is also injective. Both the source and target are $\mathbb{F}_p$-vector spaces of dimension N and it follows that the map is an isomorphism. □

Corollary 6.84. *$I = \ker(\beta)$. That is, $\ker(\beta) = E^0(BGL_p(\mathbb{F}_q))$ is principal, generated by t.*

Proof. Since $K^0 \otimes_{E^0} I \to K^0 \otimes_{E^0} \ker(\beta)$ is an isomorphism, the result follows immediately by an application Proposition 2.12. □

Proof of Theorem C. It just remains to assemble the results of this chapter. That α and β are jointly injective is Corollary 6.69. We have shown that β is surjective in Proposition 6.10 and surjectivity of α was proved in Proposition 6.32. The rational isomorphism was Proposition 6.68. The remaining results were covered in Lemma 6.73, Corollary 6.84 and Proposition 6.74. □

As a corollary to Theorem C we can give an explicit basis for $E^0(BGL_p(\mathbb{F}_q))$. Indeed, by earlier work (see Section 6.2) we have a basis B for $E^0(BT)^{\Sigma_p}$ which lifts canonically to $E^0(BGL_p(\mathbb{F}_q))$; write $\tilde{B}$ for this lift. We then have the following result.

Corollary 6.85. *The set $S = \tilde{B} \cup \{tc_p^i \mid 0 \leq i < N\}$ is a basis for $E^0(BGL_p(\mathbb{F}_q))$ over E^0.*

Proof. Since $\ker(\beta)$ is a summand in $E^0(BGL_p(\mathbb{F}_q))$ we have a decomposition

$$E^0(BGL_p(\mathbb{F}_q)) \simeq \ker(\beta) \oplus E^0(BT)^{\Sigma_p}.$$

But $\ker(\beta) = E^0(BGL_p(\mathbb{F}_q))t \simeq D^\Gamma t = E^0\{tc_p^i \mid 0 \leq i < N\}$ and $E^0(BT)^{\Sigma_p} = E^0\{B\}$. Thus, as E^0-modules,

$$E^0(BGL_p(\mathbb{F}_q)) = E^0\{tc_p^i \mid 0 \leq i < N\} \oplus E^0\{\tilde{B}\}.$$ □

Appendix A

Glossary

To ease readability of this thesis, a glossary of frequently used notation is included below.

- We use A to denote a chosen cyclic subgroup of $GL_p(\mathbb{F}_q)$ of size p^{v+1}.
- We let α denote the composition $E^0(BGL_p(\mathbb{F}_q)) \to E^0(BA) \to D$.
- For $\alpha \in J$ we let $b_\alpha = \mathrm{tr}^N_{(\mathbb{F}_q^\times)^p}(x_1^{\alpha_1} \dots x_p^{\alpha_p}) \in E^0(BN)$.
- We let β denote the restriction map $E^0(BGL_d(\mathbb{F}_q)) \to E^0(BT_d)$.
- We write c_p for the l-euler class of any subgroup of $GL_p(\overline{\mathbb{F}}_l)$; that is, c_p is the restriction of the generator σ_p of $E^0(BGL_p(\overline{\mathbb{F}}_l)) \simeq E^0[\![\sigma_1, \dots, \sigma_p]\!]$.
- We use D to denote the ring $E^0[\![x]\!]/\langle p\rangle([p^v](x))$.
- We use d to denote the generator of $E^0(B\Sigma_p) \simeq E^0[\![d]\!]/df(d)$.[1]
- We let Δ denote the diagonal subgroup of $T = T_p$ and Δ_p denote the p-part of Δ.
- We use f to denote the polynomial over E^0 for which $f(-w^{p-1}) = \langle p\rangle(w)$ in $E^0[\![w]\!]/[p](w)$.
- We let F denote the Frobenius automorphism of $\overline{\mathbb{F}}_l$ and also to denote the standard p-typical formal group law.
- We write Γ for the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. We also abuse this notation slightly to write $\Gamma = \mathrm{Gal}(\mathbb{F}_{q^p}/\mathbb{F}_q)$ where the difference is unimportant.
- In Chapter 6 we write I for the ideal of $E^0(BGL_p(\mathbb{F}_q))$ generated by t.
- We let J be the set $\{\alpha \in \mathbb{N}^p \mid 0 \leq \alpha_1 \leq \dots \leq \alpha_p < p^{nv} \text{ and } \alpha_1 < \alpha_p\}$.
- We use K to denote a finite field of characteristic coprime to p.[2]
- We write L for the extension of $\mathbb{Q} \otimes E^0$ formed by adjoining a complete set of roots of $[p^m](x)$ for each $m > 0$.
- We use l to denote a chosen prime number different to p.

[1] Not to be confused with the chosen integer greater than or equal to 1 which determines the rank of the general linear group being studied.

[2] Not to be confused with the cohomology theory K; context should ensure there is no ambiguity.

- We let N_d (or N) denote the normalizer of T_d in $GL_d(\mathbb{F}_q)$.
- We use n to denote the integer corresponding to the height of the Morava E-theory.
- We use p to denote the prime at which the Morava E-theory is localised.
- We let Φ be the group $\mathrm{Rep}(\Theta^*, GL_1(\overline{\mathbb{F}}_l)) = \mathrm{Hom}_{\mathrm{cts}}(\Theta^*, \overline{\mathbb{F}}_l^\times)$.
- We write $\psi_1,\ \psi_2$ and ψ_3 to denote the restriction maps from $E^0(BN)$ to $E^0(BT)$, $E^0(B(\Sigma_p \times \Delta))$ and $E^0(BA)$ respectively.
- We let $q = l^r$ be a power of a chosen prime number different to p.
- We write t for the unique class in $E^0(BGL_p(\overline{\mathbb{F}}_l))$ which restricts to $\prod_i [p^v](x_i)$ in $E^0(B(\overline{\mathbb{F}}_l^\times)^p) \simeq E^0[\![x_1, \ldots, x_p]\!]$.
- We let T_d (or T) denote the maximal torus of $GL_d(\mathbb{F}_q)$. Similarly, we may write $\overline{T}_d$ for the maximal torus of $GL_d(\overline{\mathbb{F}}_q)$.
- We let $\Theta = (\mathbb{Z}/p^\infty)^n$ and $\Theta^* = \mathrm{Hom}_{\mathrm{cts}}(\Theta, S^1) = \mathbb{Z}_p^n$.
- u denotes the invertible polynomial generator of E^* lying in degree -2.
- $u_1, \ldots, u_{n-1}$ denote the standard power-series generators of E^0 over $\mathbb{Z}_p$.
- We let $v = v_p(q-1)$.
- We let w denote the standard generator of $E^0(BC_p) \simeq E^0[\![w]\!]/[p](w)$.
- We let x denote the complex orientation or complex coordinate of E, or a restriction there-of.

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