Non-standard Analysis

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1 Infinitesimals

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We’ll look at two approaches: one due to Abraham Robinson (1960s) which relies on model theory, and a newer approach due to Edward Nelson (1970s) which uses an extended version of ZF set-theory.

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Other concepts from analysis can be rephrased similarly.

2 Model Theory

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In this sense, model theory has close links with formal logic.

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Starting with simple atomic formulae, one can write down rules for generating well-formed formulae inductively (e.g. 'if X and Y are wff then so is $X \vee Y$').

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Finally, a wff X which is defined in $\mathcal{M}$ is said to *be true* or *hold* in $\mathcal{M}$ if X holds when interpreted according to the maps described above.

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says addition is commutative, and

$$\begin{aligned}&\forall s,t,u,v,x,y,z\\&\quad (\; S(y,z,s) \wedge P(x,s,t) \iff P(x,y,u) \wedge P(x,z,v) \wedge S(u,v,t) \;)\end{aligned}$$

is the distributive law.

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Suppose that K is a set of sentences and X is an additional sentence which only uses constant and relation symbols already occurring in K. Then X is said to be *deducible* from K (written $K \vdash X$) if $K \cup \{\neg X\}$ is inconsistent (i.e., there is no model for $K \cup \{\neg X\}$).

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Finally, choose any constant a_B that doesn't already appear in K; then the set of sentences

$$K \cup \bigcup_{x \in \Delta_b} \{B(x, a_B)\}$$

is called the *enlargement of K (with respect to B).*

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Corollary 2.3.

Let K be a consistent set of sentences and let B be a concurrent binary relation in K. Then the enlargement of K has a model.

3 Robinson's Non-standard Analysis

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The enlargement of K with respect to B has a model by (2.2). Any such model is called a *non-standard model of analysis*.

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  holds for $\mathbb{R}$ so must hold for $*\mathbb{R}$. In particular, given any unlimited y there is an element $\varepsilon = \frac{1}{y} \in *\mathbb{R}$.

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- By construction, there exists an element $y \in *\mathbb{R}$ bigger than every standard real number. We'll call such an element *unlimited.*
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- Notice that 0 counts an an infinitesimal.

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4 Internal Set Theory

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We'll write $\forall^{\text{st}}$ for 'for all standard', and similarly for $\exists^{\text{st}}$.

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Let $A = A(x, t_1, \ldots, t_n)$ be an internal formula with free variables $x, t_1, \ldots, t_n$. Then the axiom of *transfer* is

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Since $\neg\forall$ is the same as $\exists\neg$, we get a dual form for the transfer axiom which, roughly speaking, says $\exists x A(x) \implies \exists^{\text{st}} x A(x)$. Hence any set which is uniquely determined by an internal formula is necessarily standard (e.g. $\emptyset,\ 0,\ 1,\ \mathbb{N},\ \mathbb{R}, \ldots$).

Aside: finite sets

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Let $B(x, y)$ be an internal formula with free variables x and y. Then the axiom of *idealization* is

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Informally, a relation B is simultaneously satisfiable on every standard finite set if and only if it is simultaneously satisfiable for all standard sets.

It turns out that a set z is standard finite iff every element of z is standard. Thus, idealization corresponds to the role played by concurrent relations in Robinson's enlargements.

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As the predicate st is not included in ZFC, the correct interpretation in our extended set theory is that $C(z)$ must be an internal formula. That is, the axiom isn't valid if used with external formulas.

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I think the correct comparison with Robinson's work is the following: given a (possibly external) relation S in $*\mathbb{R}$, restricting to $\mathbb{R}$ will give a relation in the original structure $\mathcal{M}$, which will then correspond to a standard internal relation in $*\mathbb{R}$.

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- The existence of a finite set as above is useful in non-standard analysis proofs, as finite sets have nice properties.
- Nelson's paper has a 'grab-bag of non-standard analysis' which gives a good introduction to the kind of methods that can be used in his non-standard analysis.

5 References

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